Courant Institute of Mathematical Sciences

MAGNETO-FLUID DYNAMICS DIVISION

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ASYMPTOTIC DESCRIPTION OF THE CUSPS OF A HYDROMAGNETIC FIGURE OF EQUILIBRIUM

Robert E. Dowd

1 February 1964

AEC Research and Development Report

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ABSTRACT

The topic of this work is suggested by the Atomic Energy Commission's controlled fusion research, which involves an attempt to confine a perfectly conducting plasma in a strong magnetic field. Of several confinement schemes proposed, the cusped geometry model offers promise of stability in exchange for some leakage through the cusps. It is the purpose of this dissertation to provide asymptotic formulas for the cusps of the simplest cusped geometry, which is a spindle-shaped figure consisting of two tapered circular cones set back to back. This figure thus has one circular knife-edged cusp where the cones are joined and two identical axisymmetric point cusps. The spindle is supposed to contain a plasma which is considered a fluid at rest and therefore at constant pressure. An external vacuum magnetic field due to a quadrupole at infinity balances the fluid pressure. The problem of finding the shape of the interface between the fluid and the vacuum is stated as a differential equation with two boundary conditions. Garabedian, by analytic continuation into the four-dimensional space of two complex variables, restates this problem as an integral equation in terms of an analytic function $g(z)$. In the real plane the trace of the interface is given by $\bar{z} = g(z)$.

At the edge cusp, $z = z_1$, conformal mapping is used to obtain the convergent series expansion of $g(z)$

$$\bar{z} = g(z) = -z + iC(z-z_1)^{\frac{3}{2}} + \sum_{n=4}^{\infty} a_n (z-z_1)^{\frac{n}{2}}$$

At the point cusp $z = z_0$ we have a branch point which greatly complicates matters. We extend work of Lewy to cover this singular case, obtaining thereby an asymptotic expansion

$$\bar{z} = g(z) = z + a_0 i(z-z_0)^{\frac{3}{2}}\left\{\frac{1}{\ln(z-z_0)} + \frac{2\ln(-\ln(z-z_0))}{(\ln(z-z_0))^2} - \frac{\left(-\frac{9}{4} + 2\ln\frac{a_0}{4}\right)}{(\ln(z-z_0))^2} + 0\left(\frac{(\ln(-\ln(z-z_0)))^2}{(\ln(z-z_0))^3}\right)\right\}$$

valid in the neighborhood of a point cusp.

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INTRODUCTION

The topic of this work is suggested by the Atomic Energy Commission's controlled fusion research [4], which involves an attempt to confine a perfectly conducting plasma in a strong magnetic field. Several confinement schemes have been proposed. Of these, the cusped geometry and picket fence models [2, 4, 12] offer promise of a stable figure of equilibrium in exchange for a certain amount of leakage through the cusps. Among the matters of interest in connection with these cusped models is the question of the asymptotic shape of the cusps for which, unfortunately, no analytic description is available. It is the purpose of this dissertation to remedy that deficiency by providing asymptotic formulas for the cusps of the simplest cusped geometry, which is the spindle-shaped configuration of figure 1.

This axisymmetric configuration consists of two identical tapered cones whose broad ends are joined to form a circular knife-edged cusp, generally called the ring, or edge, cusp. The two mirror image pointed ends of the resultant spindle-shaped figure are axisymmetric point cusps. This figure is supposed to contain a fully ionized plasma which we regard as a perfectly conducting fluid continuum at rest, and therefore at constant pressure. Our spindle-shaped figure results from a balance between this fluid pressure and the force of an external vacuum magnetic field due to a quadrupole at infinity.

The problem of finding the shape of the interface between the fluid and the surrounding vacuum is stated mathematically as a differential equation with two boundary conditions, expressed in terms of a potential ψ, analogous to the stream function of fluid dynamics. The second boundary condition would be superfluous if the boundary were given, but because the boundary is unknown, we are faced with a free boundary problem which requires an extra boundary condition.

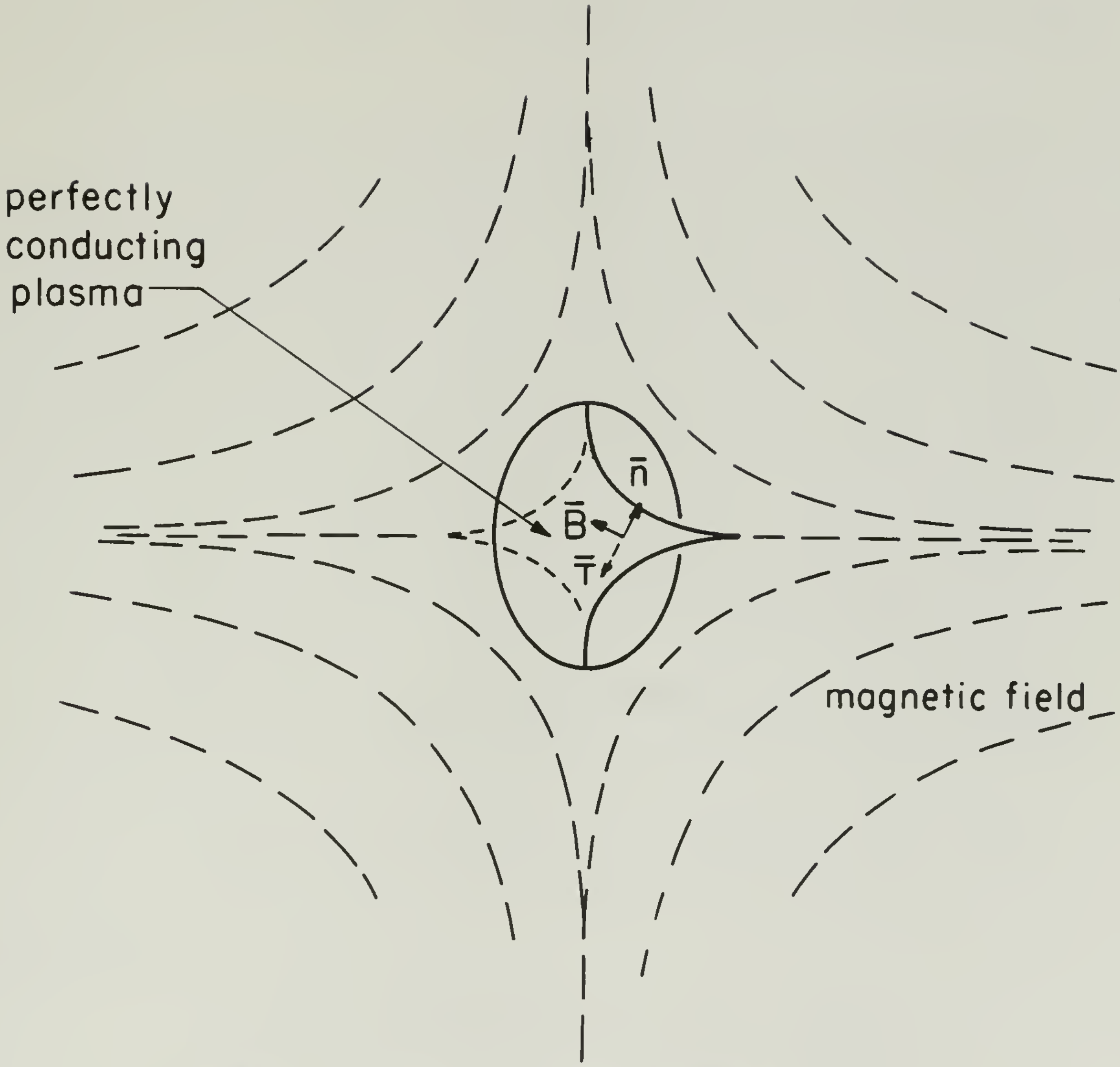


Figure I. Plasma Spindle in Magnetic Field.

This problem is so difficult that in spite of long consideration it remains unsolved. Not only is the problem difficult, but its formulation in terms of a differential equation in ψ is particularly awkward because the equation contains a singular coefficient $1/y$ that becomes infinite at the point cusps.

We wish to find asymptotic expressions for the cusps because of the problem of leakage of the plasma through these cusps. To do so we shall utilize the work of Garabedian [10] who, by analytic continuation into the four-dimensional space of two complex variables has replaced the differential equation for ψ with a more amenable integral equation for an unknown analytic function $g(z)$. The integral equation is obtained with the aid of a Riemann function. In the real plane the integral equation defines ψ, and the trace of the spindle boundary yields a pattern consisting of four mirror image curves Γ whose equations are given in the form $\bar{z} = g(z)$. In order to satisfy the integral equation it is necessary to choose the coefficients and exponents of $g(z)$ in a particular way.

At the edge cusp $z = z_1$ we invoke Schwarz's reflection principle to obtain a convergent series expansion of $g(z)$. Thus

$$\bar{z} = g(z) = -z + i\, C(z-z_1)^{\frac{3}{2}} + \sum_{n=4}^{\infty} a_n (z-z_1)^{\frac{n}{2}}$$

which becomes

$$x = \frac{-C_3}{\sqrt{2}} (y-y_1)^{\frac{3}{2}} - \frac{3C_3^{\,2}}{4} (y-y_1)^2 + 0((y-y_1)^{\frac{5}{2}})$$

in terms of x and y.

At the point cusp $z = z_0$ the contour integral which represents ψ in terms of the Riemann function is greatly complicated by the presence of a branch point. Lewy [15] has treated some similar problems from a different

standpoint. He gives a general expansion which suggests the form of g(z). His expansion is valid for geometries quite similar to ours but is not quite adequate here because of the unusually singular nature of our point cusp. Nevertheless Lewy's work serves as a starting point from which we ultimately obtain an asymptotic expansion of g(z) at the point cusp. With sufficient care given to the branch point, the asymptotic expansion of the point cusp is found to be

$$\bar{z} = g(z) = z + a_0 i(z - z_0)^{\frac{3}{2}} \left\{ \frac{1}{\ell n(z - z_0)} + \frac{2\ell n(-\ell n(z - z_0))}{(\ell n(z - z_0))^2} - \frac{(\frac{-9}{4} + 2\ell n \frac{a_0}{4})}{(\ell n(z - z_0))^2} + 0\left(\frac{[\ell n(-\ell n(z - z_0))]^2}{(\ell n(z - z_0))^3} \right) \right\}$$

In terms of x and y this becomes

$$y = \frac{a_0(x - x_0)^{\frac{3}{2}}}{2} \left\{ \frac{-1}{\ell n(x - x_0)} - \frac{2\ell n(-\ell n(x - x_0))}{(\ell n(x - x_0))^2} + \frac{(\frac{-9}{4} + 2\ell n \frac{a_0}{4})}{(\ell n(x - x_0))^2} + 0\left(\frac{[\ell n(-\ell n(x - x_0))]^2}{(\ell n(x - x_0))^3} \right) \right\}$$

The expansions of g(z) at the cusps constitute the principal contribution of this dissertation.

1. PLASMA SUSPENDED IN AN AXISYMMETRIC MAGNETIC FIELD

The physical situation of figure 1 is that of a fully ionized gas suspended in a perfect vacuum by means of a steady magnetic field. This field is due to a quadrupole at infinity. The boundary, or interface, separating the vacuum from the plasma is an axially symmetric spindle-shaped figure, of the type commonly referred to as "cusped geometry". It is the simplest such cusped geometry, due solely to the magnetic vector $\bar{B}$ and the plasma itself. That is, there is no electric field ($\bar{E} = \bar{D} = 0$) and no charge distribution ($\rho = 0$). In the vacuum we may put the magnetic permeability $\mu = 1$ and, away from the infinite quadrupole, $\bar{B}$ is derivable from a scalar potential ϕ. More explicitly, we have

$$\bar{B} = -\nabla \phi$$

The boundary consists of lines of force of $\bar{B}$ so that, if $\bar{n}$ is the outward normal,

(1.1) $$\bar{B} \cdot \bar{n} = -\frac{\partial \phi}{\partial n} = 0$$

on this boundary.

Let z denote distance along the axis of symmetry and r denote distance radially outward from this axis. Then at infinity, i.e., near the quadrupole, we find [18]

(1.2) $$\phi \approx 2z^2 - r^2$$

Because of Maxwell's equations we obtain [1, 17]

(1.3) $$\nabla^2 \phi = -\nabla \cdot \bar{B} = 0$$

throughout the vacuum.

The plasma is assumed to be perfectly conducting fluid at rest. A fluid continuum at rest can not support internal pressure gradients. Therefore the pressure is constant throughout the fluid and also at the interface. Because the plasma is a perfect conductor any current arising from ionization will be displaced instantly to the surface so that there will be only surface current.

To find the surface force due to the magnetic field we use Maxwell's stress tensor [1, 13, 17], written in dyadic form. For $\overline{E} = 0$ this dyadic is [13]

$$\overleftrightarrow{T} = \frac{1}{4\pi}\left\{\overline{B}\overline{B} - \frac{1}{2}\overleftrightarrow{I}(\overline{B}\cdot\overline{B})\right\}$$

where $\overleftrightarrow{I}$ is the identity dyadic. The magnetic force per unit area on the interface is

$$\overline{F} = \bar{n}\cdot\overleftrightarrow{T} = \frac{1}{4\pi}\left\{-\frac{(\overline{B}\cdot\overline{B})}{2}\right\}\bar{n}$$

where we have used $\bar{n}\cdot\overline{B} = 0$, which results from the orthogonality of $\bar{n}$ and $\overline{B}$. This means that there is magnetic pressure which may be expressed as

$$-\overline{F}\cdot\bar{n} = \frac{(\overline{B}\cdot\overline{B})}{8\pi} = \frac{(\nabla\phi\cdot\nabla\phi)}{8\pi}$$

In the steady state the fluid pressure p, which is constant, must balance the magnetic force $(-\overline{F}\cdot\bar{n})$ at the interface. This gives

$$p = -\overline{F}\cdot\bar{n} = \frac{(\nabla\phi\cdot\nabla\phi)}{8\pi}$$

or

(1. 4) $$(\nabla\phi\cdot\nabla\phi) = 8\pi p = \text{constant}$$

If the shape of the boundary were known, our problem would be to solve Laplace's equation for ϕ in the vacuum and we would need only one boundary condition. However, our problem is also to find the shape of the boundary. Therefore what we have is a free boundary problem, which requires a second boundary condition. The full mathematical statement of the problem thus asks for a solution of

$$\nabla^2 \phi = 0 \tag{1.3}$$

subject to the boundary conditions

$$\nabla \phi \cdot \bar{n} = 0 \tag{1.1}$$

and

$$(\nabla \phi \cdot \nabla \phi) = 8\pi p = \text{constant} \tag{1.4}$$

This formulation also describes the free boundary problem of a vapor-filled cavity in steady flow of water. There ϕ is the velocity potential, p is the vapor pressure which is constant, and the boundary conditions arise from Bernoulli's theorem and the assumption that water does not enter the cavity. [3]

Were we concerned here with the hydrodynamic problem, we would consider the possibility of restating the problem in terms of the stream function ψ instead of the velocity potential ϕ. Of course our plasma problem is now a mathematical one and we are free to introduce ψ simply as a convenient function. Therefore we define the point function ψ which satisfies equation (1.3) identically by

$$\frac{\partial \phi}{\partial r} = \frac{-1}{r} \frac{\partial \psi}{\partial z} \quad , \quad \frac{\partial \phi}{\partial z} = \frac{1}{r} \frac{\partial \psi}{\partial r} \tag{1.5}$$

As we will soon see, ψ is constant on lines of force of the field of $\overline{B}$.

Let n and s denote distance in a meridian plane along the outward normal and along the surface, respectively. Then (1. 5) becomes

$$\frac{\partial\phi}{\partial n} = \frac{-1}{r}\frac{\partial\psi}{\partial s} \qquad \frac{\partial\phi}{\partial s} = \frac{1}{r}\frac{\partial\psi}{\partial n} \tag{1. 6}$$

Because the curl of a gradient is zero we have

$$0 = \nabla x \nabla \phi = \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial^2\psi}{\partial z^2} - \frac{1}{r^2}\frac{\partial\psi}{\partial r} \tag{1. 7}$$

The boundary condition (1. 1) is

$$0 = \nabla\phi\cdot\bar{n} = -\frac{1}{r}\left(\frac{\partial\psi}{\partial s}\bar{n} - \frac{\partial\psi}{\partial n}\bar{s}\right).\ \bar{n} = \frac{-1}{r}\frac{\partial\psi}{\partial s}$$

This equation holds for any line of force of the field of $\bar{B}$, which tells us that ψ is constant on each such line. In particular

$$\psi = \text{constant} \tag{1. 8}$$

on the boundary. Substituting (1. 6) into (1. 4) we obtain

$$\text{constant} = \nabla\phi\cdot\nabla\phi = \left\{-\frac{1}{r}\left(\frac{\partial\psi}{\partial s}\bar{n} - \frac{\partial\psi}{\partial n}\bar{s}\right)\right\}^2 = \frac{1}{r^2}\left[\left(\frac{\partial\psi}{\partial s}\right)^2 + \left(\frac{\partial\psi}{\partial n}\right)^2\right]$$

$$= \left(\frac{1}{r}\frac{\partial\psi}{\partial n}\right)^2$$

so that

$$\frac{1}{r}\frac{\partial\psi}{\partial n} = \text{constant} \tag{1. 9}$$

on the boundary. Equations (1. 2) and (1. 5) tell us that, at infinity, ψ is of the form zr^2. We may sum up this analysis by re-expressing our mathematical problem as

(1.7) $$\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = 0$$

with the boundary conditions

(1.8) $$\psi = \text{constant}$$

and

(1.9) $$\frac{1}{r}\frac{\partial \psi}{\partial n} = \text{constant}$$

This problem, which has thus far defied solution, will not be solved here. What we propose to do is to find the asymptotic shape of the boundary in the vicinity of the edge cusp, where z is arbitrarily small, and the point cusps, where r is arbitrarily small. By utilizing the smallness of one of the two variables we can discard certain terms and simplify the problem. To make such a procedure useful we first convert our differential equation to a complex integral equation. In this manner the free boundary problem described by a partial differential equation and boundary conditions for a potential function ψ becomes a free boundary problem described by an integral equation for a complex analytic function g(z) satisfying a single boundary condition. The desired asymptotic descriptions of the cusps result from a delicate analysis of the free boundary problem for g(z).

2. EQUIVALENCE OF THE REAL DIFFERENTIAL EQUATION AND A COMPLEX INTEGRAL EQUATION

Garabedian's Extension to the four-dimensional space of two complex variables

Rather than refer specifically to either the plasma or hydrodynamic problem described above, we shall simply refer to the mathematical problem which represents both. The formulation in terms of ψ will be used, but in cylindrical coordinates x and y, where the x-axis is the axis of symmetry of the cusped figure and y denotes distance radially outward from this axis. Then our equation (1. 7) becomes

(2. 1) $$\psi_{xx} + \psi_{yy} - \frac{1}{y}\,\psi_y = 0$$

with

(2. 2) $$\psi = 0 \quad , \quad \frac{1}{y}\,\frac{\partial\psi}{\partial n} = 1$$

on the boundary. The constants 0 and 1 are chosen arbitrarily, but without loss of generality, because ψ is an unscaled potential to which an arbitrary constant can always be added. Here n denotes the outward normal to the boundary. At infinity, ψ is of the form xy^2.

If the condition at infinity is dropped and the free boundary is prescribed, our problem becomes a Cauchy problem. Even from this simplified point of view, however, the attainment of a complete solution is difficult. What we propose to do is expand ψ asymptotically in the neighborhood of the cusps, thereby getting asymptotic expressions for the cusp contours. The asymptotic approach we wish to use for small x at one cusp and small y at the other is not easily applied to equation (2. 1) because we do not have any way of knowing the magnitude of the derivatives of ψ for small x or y. Moreover the differential equation (2. 1) is extremely difficult as it contains the term $1/y$ which becomes infinite at the point cusps. Garabedian [10] simplifies the problem

considerably by analytic continuation into the four-dimensional space of two complex variables (z, z^*). Here the elliptic equation (2.1) in (x, y) becomes a hyperbolic equation in (z, z^*). Garabedian [10] then replaces the difficult differential equation by a much simpler integral equation found by solving the Cauchy problem in terms of a Riemann function. This expresses the completely unknown function ψ in terms of an analytic function $g(z)$ whose order of magnitude is known.

To begin with let us consider the new variables

$$z = z + iy \quad , \quad z^* = x - iy$$

where x and y are themselves complex. Then, clearly,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial z^*}\frac{\partial z^*}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial}{\partial z^*}\frac{\partial z^*}{\partial y} = i\left\{\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right\}$$

whence

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad , \quad \frac{\partial}{\partial z^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial z^*}$$

These expressions are substituted into equations (2.1) and (2.2) to yield

(2.3) $$\psi_{zz^*} + \frac{1}{2(z-z^*)}\left(\psi_z - \psi_{z^*}\right) = 0$$

with

(2.4) $$\psi = 0 \quad , \quad \frac{2i}{(z-z^*)}\frac{\partial\psi}{\partial n} = 1$$

on the boundary.

Note that the characteristics of equation (2. 3) are the complex surfaces z = constant and z^* = constant. It is well known that the solution to an elliptic equation is analytic. Therefore ψ, as the solution of (2.1) is an analytic function of x and y. We have defined $x = \frac{1}{2}(z + z^*)$ and $y = \frac{1}{2i}(z - z^*)$ which are analytic functions of z and z^*. Therefore $\psi(z, z^*)$ is an analytic function of z and z^* and may be continued analytically into a neighborhood of the boundary in the space of two complex variables z and z^*.

Garabedian [10] observes that, for certain regions, equation (2. 3) admits a formal solution in integral form. This type of solution employs as kernel a function known as the Riemann function [8, 9, 10]. The Riemann function [8, 9, 12] of our equation is

$$R(z, z^*; t, t^*) = \frac{\sqrt{(z - t^*)(t - z^*)}}{t - t^*} F\left(\frac{(z - t)(z^* - t^*)}{(z - t^*)(z^* - t)}\right) \tag{2.5}$$

which satisfies (2. 3) as a function of (z, z^*) and the adjoint of (2. 3) as a function of (t, t^*). In addition

$$\frac{\partial R}{\partial z} = \frac{1}{2(z - z^*)} R \quad ; \quad \frac{\partial R}{\partial t} = \frac{-R}{2(t - t^*)} \tag{2.6}$$

on the characteristic z^* = constant,

$$\frac{\partial R}{\partial z^*} = \frac{-1}{2(z - z^*)} R \quad ; \quad \frac{\partial R}{\partial t^*} = \frac{1}{2(z - t^*)} R \tag{2.7}$$

on the characteristic z = constant, and

$$R(z, z^*; z, z^*) = 1 \tag{2.8}$$

The function $F(w)$, given by

$$F(w) = F(-\frac{1}{2}, -\frac{1}{2}; 1; w) = \sum_{k=o}^{\infty} \frac{[1 \cdot 3 \cdot 5 \cdots (2k-3)]^2 w^k}{2^{2k} (k!)^2}$$

is the hypergeometric series with parameters $(-\frac{1}{2}, -\frac{1}{2}, 1)$ and satisfies the ordinary differential equation

$$w(1-w)F''(w) + F'(w) - \frac{F(w)}{4} = 0$$

Before we can use the Riemann function to obtain a solution of (2. 3) in the four-dimensional space of two complex variables (z, z^*), we must show that both Stokes' theorem in a special form, and Riemann's representation formula for the Cauchy problem, hold in this space. As Stokes' theorem is required in the derivation of Riemann's formula, we shall obtain the required form of Stokes' theorem first.

Special case of Stokes' theorem

Let G be any smooth two-dimensional surface whose boundary is Γ_1 and let $u(z, z^*)$ and $v(z, z^*)$ be two complex analytic functions of the two complex variables z and z^* in the four-dimensional space defined by these two complex variables. Then, as we shall prove, the following version of Stokes' theorem holds:

$$(2.9) \quad \iint_G \left[\frac{\partial}{\partial z} u(z, z^*) + \frac{\partial}{\partial z^*} v(z, z^*)\right] dz dz^* = \int_{\Gamma_1} [u(z, z^*)dz^* - v(z, z^*)dz]$$

The four-dimensional space may be described by z, $\bar{z}$, z^* and $\overline{z^*}$. Then, according to the calculus of exterior differential forms (cf. Buck [6]) we are able to express the contour integrals of u and v as surface integrals of du and dv,

$$\int_{\Gamma_1} [u(z, z^*)dz^* - v(z, z^*)dz] = \iint_G (du\, dz^* - dv\, dz)$$

$$= \iint_G (\frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial \bar{z}} d\bar{z} + \frac{\partial u}{\partial z^*} dz^* + \frac{\partial u}{\partial \overline{z^*}} d\overline{z^*})\, dz^*$$

$$- \iint_G (\frac{\partial v}{\partial z} dz + \frac{\partial v}{\partial \bar{z}} d\bar{z} + \frac{\partial v}{\partial z^*} dz^* + \frac{\partial v}{\partial \overline{z^*}} d\overline{z^*})\, dz$$

Because of the conventions

$$(dz)^2 = (dz^*)^2 = 0 \;\; ; \;\; dz\,dz^* = -dz^*dz$$

and the Cauchy-Riemann equations [5]

$$\frac{\partial u}{\partial \bar{z}} = 0 \;\; ; \;\; \frac{\partial u}{\partial \overline{z^*}} = 0 \;\; ; \;\; \frac{\partial v}{\partial \bar{z}} = 0 \;\; ; \;\; \frac{\partial v}{\partial \overline{z^*}} = 0$$

this becomes

$$(2.9) \quad \iint_G (\frac{\partial}{\partial z} u(z,z^*) + \frac{\partial}{\partial z^*} v(z, z^*))\, dz\, dz^* = \int_{\Gamma_1} (u(z, z^*)\, dz^* - v(z, z^*)dz)$$

which is the desired theorem. The result is seen to combine the more usual form of Stokes' theorem with the Cauchy integral theorem.

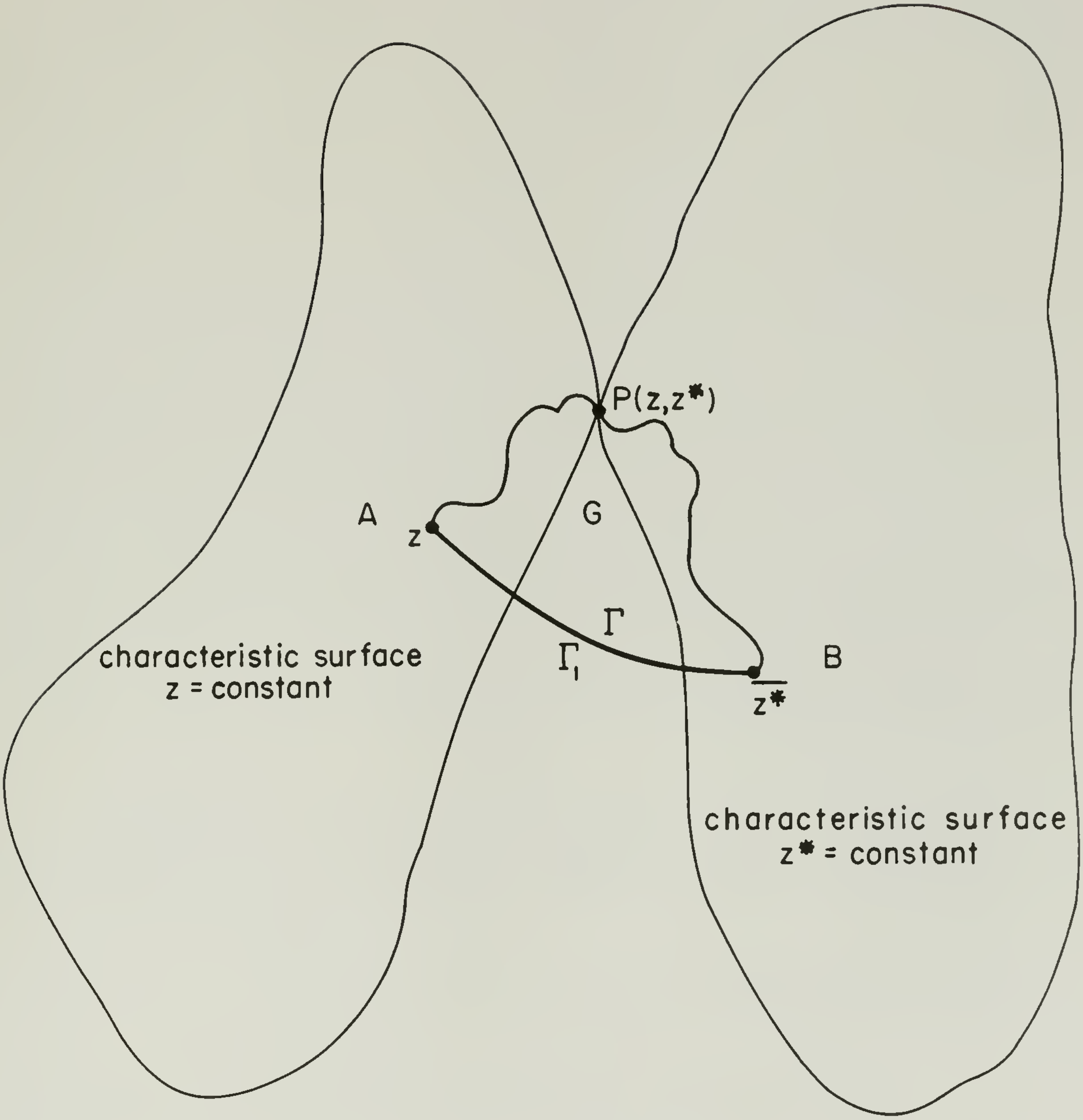


Figure 2. Schematic View of Contour in Four-Dimensional Space of two Complex Variables.

Proof of Riemann's representation theorem

Consider a portion of the intersection of the cusped spindle of plasma with a meridian plane and denote the arc of its boundary running from $\bar{z}^*$ to z by Γ. For convenience we will refer to $\bar{z}^*$ and z as B and A respectively, as indicated in figure 2. In the four-dimensional space of two independent complex variables (z, z^*) the characteristic plane z = constant which passes through z and the characteristic plane z^* = constant which passes through $\bar{z}^*$ will intersect in a single point $P(z, z^*)$. Connect P and A by means of any differentiable curve lying in the plane z = constant and connect B and P by means of any differentiable curve lying in the plane z^* = constant. Then pass a smooth surface through ABPA. We will refer to the closed curve ABPA as Γ_1, and to the smooth surface which it bounds as G.

If we multiply equation (2.3) by ω and its adjoint equation by ψ and subtract one from the other, and then integrate the result over G, with (t, t^*) as running variables of integration instead of (z, z^*), we get

$$(2.10) \qquad -\iint_G \left\{ \omega L(\psi) - \psi L^*(\omega) \right\} dt\, dt^* = \iint_G \left\{ \frac{\partial}{\partial t} \left(\psi \frac{\partial \omega}{\partial t^*} - \frac{1}{2(t - t^*)} \psi \omega\right) - \frac{\partial}{\partial t^*} \left(\omega \frac{\partial \psi}{\partial t} - \frac{1}{2(t - t^*)} \psi\omega\right) \right\} dt\, dt^*$$

Because $L(\psi) = 0$ and $L^*(\omega) = 0$ this integral is equal to zero.

Set

$$u = \psi \frac{\partial \omega}{\partial t^*} - \frac{1}{2(t - t^*)} \psi\omega$$

$$v = -\omega \frac{\partial \psi}{\partial t} + \frac{1}{2(t - t^*)} \psi\omega$$

Whenever u and v as defined above are analytic in G and on Γ_1 we can apply Stokes' theorem to (2.10). We already know that $\psi(t, t^*)$ is an analytic function of (t, t^*) but we have not defined ω. Let $\omega = R(z, z^*; t, t^*)$ be the Riemann function given by equation (2.5). Then ω is analytic except for poles and branch points that occur at $t = t^*$, $t = z^*$, and $t^* = z$.

We know that at most one point of Γ touches the real axis $t = \bar{t}$. Suppose that B is displaced along Γ to slightly above the real axis. Then no point of Γ touches this axis. The point P at which $t = t^* = z = z^*$ lies in the surface $t = t^*$ as does the axis of symmetry. As we choose the paths AP and PB arbitrarily, we will choose them so they do not intersect the surface $t = t^*$, except at P. Thus no point of Γ_1 or G touches the real axis, and at no point except P do we have $t = t^*$. At A we have $t = z$ so that for $t^* = \bar{t}$ to equal z on Γ is impossible without crossing the axis of symmetry. Similarly, at B we have $\bar{t} = z^*$ so that it is impossible to have $t = z^*$ on Γ without crossing the axis of symmetry.

Because $t^* = z^*$ on PB and $t = z$ on AP we do not have $t = z^*$ or $t^* = z$ on these curves except where $t = t^*$ which is only at P. The function $\omega = R$ is analytic and single-valued at P and clearly has no singularities at any other point of Γ_1 or G. Therefore $R = \omega$ can be considered analytic on Γ_1 and G and we may apply Stokes' theorem (2.9) to (2.10) which becomes

$$(2.12) \qquad 0 = \iint_G \left\{ \frac{\partial}{\partial t} \left(\psi \frac{\partial R}{\partial t^*} - \frac{1}{2(t - t^*)} \psi R\right) - \frac{\partial}{\partial t^*}\left(R \frac{\partial \psi}{\partial t} - \frac{1}{2(t - t^*)} \psi R\right) \right\} dt\, dt^*$$

$$= \int_{\Gamma_1} \left\{ \left(\psi \frac{\partial R}{\partial t^*} - \frac{1}{2(t - t^*)} \psi R\right) dt^* + \left(R \frac{\partial \psi}{\partial t} - \frac{1}{2(t - t^*)} \psi R\right) dt \right\}$$

We may then rewrite (2.12) as

$$(2.13)\qquad \int_{\Gamma}\left\{R\left(\frac{\partial\psi}{\partial t}-\frac{1}{2(t-t^*)}\psi\right)dt+\psi\left(\frac{\partial R}{\partial t^*}-\frac{1}{2(t-t^*)}R\right)dt^*\right\}$$

$$+\int_{PB} R\left(\frac{\partial\psi}{\partial t}-\frac{1}{2(t-t^*)}\psi\right)dt+\int_{AP}\psi\left(\frac{\partial R}{\partial t^*}-\frac{1}{2(t-t^*)}R\right)dt^*=0$$

where we have made use of the facts that $dt = 0$ on AP and $dt^* = 0$ on PB because these are segments of characteristics on which t = constant and t^* = constant respectively. Moreover, by (2.7) we see that the last integral in (2.13) is zero, and because $\psi = 0$ on Γ the last three terms in the first integral of (2.13) vanish. The remaining integrals are all defined because of the analyticity of R and ψ.

Consider the contour integral

$$I_1=\int_{PB} R\left(\frac{\partial\psi}{\partial t}-\frac{1}{2(t-t^*)}\psi\right)dt=(R\psi)_P-(R\psi)_B-\int_{PB}\psi\left(\frac{\partial R}{\partial t}+\frac{R}{2(t-t^*)}\right)dt$$

The last integral on the right side of this equation vanishes by (2.6). Because R is defined at B and $\psi = 0$ on Γ we find $(R\psi)_B = 0$. At P, $R = 1$ by (2.8) so that

$$(2.14)\qquad I_1=\psi(z,z^*)$$

Because $\psi = 0$ on Γ it is evident that

$$(2.15) \qquad 0 = \frac{1}{2} \int_{\Gamma} R\left(\frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial t^*} dt^*\right)$$

or

$$\frac{1}{2} \int_{\Gamma} R \frac{\partial \psi}{\partial t} dt = -\frac{1}{2} \int_{\Gamma} R \frac{\partial \psi}{\partial t^*} dt^*$$

When (2.15) and (2.14) are substituted in (2.13) we obtain

$$(2.16) \qquad \psi(z, z^*) = -\frac{1}{2} \int_{\Gamma} R\left(\frac{\partial \psi}{\partial t} dt - \frac{\partial \psi}{\partial t^*} dt^*\right)$$

The curve Γ lies in the real domain $t^* = t$ so that we may speak meaningfully of such things as $ds = |dt|$, the differential of arc length along Γ. Now we wish to let B, the lower end of Γ, approach the real axis. Here ψ and its derivatives are single-valued and R is identically zero. Therefore equation (2.16) remains valid as B approaches the real axis. At any point of Γ let $\bar{n}$ be the unit outward normal and $\bar{s}$ be the unit tangent. Moreover let $\bar{i}_1$ and $\bar{i}_2$ be unit vectors in the x and y directions respectively. Then

$$\bar{s} = \frac{dx}{ds} \bar{i}_1 - \frac{dy}{ds} \bar{i}_2$$

$$\bar{n} = \frac{dy}{ds} \bar{i}_1 + \frac{dx}{ds} \bar{i}_2$$

$$\nabla \psi = \frac{\partial \psi}{\partial x} \bar{i}_1 + \frac{\partial \psi}{\partial y} \bar{i}_2$$

From these relationships we obtain

$$\frac{\partial\psi}{\partial n} = \nabla\psi \cdot \bar{n} = \frac{\partial\psi}{\partial x}\frac{dy}{ds} + \frac{\partial\psi}{\partial y}\frac{dx}{ds}$$

which, with

$$\left(\frac{\partial\psi}{\partial t}\frac{dt^*}{|dt|} - \frac{\partial\psi}{\partial t^*}\frac{dt}{|dt|}\right) = -i\left(\frac{\partial\psi}{\partial x}\frac{dy}{ds} + \frac{\partial\psi}{\partial y}\frac{dx}{ds}\right)$$

is substituted in (2.16) to give

(2.17) $$\psi(z, z^*) = \frac{1}{2i}\int_{\Gamma} R\frac{\partial\psi}{\partial n}\ |dt|$$

Our free boundary condition on Γ is

$$\frac{\partial\psi}{\partial n} = \frac{(t - t^*)}{2i}$$

When this is put into (2.17) it yields

(2.18) $$\psi(z, z^*) = \frac{1}{4i}\int_{\Gamma}\sqrt{(z - \bar{t})(z^* - t)}\ F\left(\frac{(z - t)(z^* - \bar{t})}{(z - \bar{t})(z^* - t)}\right)|dt|$$

This is Riemann's representation formula for our problem. The result, without the detailed proof, is given in [10].

Equivalence of the real and complex equations

Equation (2.18) defines ψ on a two-dimensional surface through Γ in the four-dimensional space of z and z*. We wish to continue the right side of (2.18) analytically to specify $\psi(z, z^*)$ for all values of z and z* in a four-dimensional neighborhood of Γ. Reference [11] proves that the free boundary Γ is analytic and that it is described by the equation

$$(2.19) \qquad \bar{z} = g(z)$$

where g(z) is analytic. Then x and y are taken as analytic functions of arc length s so that

$$z = x(s) + iy(s) \quad , \quad g(z) = x(s) - iy(s)$$

and

$$\frac{dg}{ds}\,\frac{dz}{ds} = \frac{dg}{dz}\left(\frac{dz}{ds}\right)^2 = (x'(s))^2 + (y'(s))^2 = 1$$

along Γ. More explicitly

$$ds = \sqrt{g'(z)}\, dz$$

on Γ. This is substituted into (2.18), along with $\overline{z^*} = \bar{g}(z^*)$, to give

$$(2.20) \qquad \psi(z, z^*) = \frac{1}{4i} \int_{\bar{g}(z^*)}^{z} \sqrt{(z - g(t))(z^* - t)g'(t)}\; F\left(\frac{(z-t)(z^* - g(t))}{(z - g(t))(z^* - t)}\right) dt$$

For all $g(z)$ of practical interest any branch point of the integrand of (2. 20) will lie outside of the region in which the path of integration lies. Then, for each fixed $\overline{z^*}$ on Γ both sides of (2. 20) are analytic functions of z identical on Γ and therefore identical in a neighborhood of Γ. For each fixed z both sides of (2. 20) are analytic functions of z^* which have the same values for $\overline{z^*}$ on Γ and therefore have the same values for all z^*. Thus by analytic continuation, formula (2. 20) holds identically for all values of z and z^*.

In the real plane $z^* = \bar{z}$

$$\psi(z, \bar{z}) = \frac{1}{4i} \int_{\overline{g(z)}}^{z} \sqrt{(z - g(t))(\bar{z} - t)\, g'(t)}\; F\left(\frac{(z - t)(\bar{z} - g(t))}{(z - g(t))(\bar{z} - t)}\right) dt$$

For z_0 any point of Γ the above integral may be written

$$I = \int_{\overline{g(z)}}^{z} = \int_{z_0}^{z} - \int_{\overline{g(z_0)}}^{\overline{g(z)}} = I_1 - I_2$$

In I_2 set $w = g(t)$. Then $z = \bar{g}(\bar{z}) = \bar{g}(g(z))$ on Γ and by analytic continuation we can invert to obtain $t = \bar{g}(w)$ and $\bar{g}'(w) g'(t) = 1$. Substituting into I_2, we find $\overline{I_2} = I_1$. This shows that $\psi(z, \bar{z})$ = real and that the simpler formula

$$\psi(z, \bar{z}) = \mathrm{Re}\left\{ \frac{1}{2i} \int_{z_0}^{z} \sqrt{(z - g(t))(\bar{z} - t) g'(t)}\; F\left(\frac{(z - t)(\bar{z} - g(t))}{(z - g(t))(\bar{z} - t)}\right) dt \right\}$$

is valid for z_0 on Γ. Note that $\psi = 0$ on Γ because

$$I_0 = \int_{z_0}^{z} \sqrt{(\bar{z} - t)(z - g(t))g'(t)} \quad F\left(\frac{(z - t)(\bar{z} - g(t))}{(z - g(t))(\bar{z} - t)}\right) dt = \text{real}$$

there. For a known analytic boundary Γ this equation generates solutions of the Cauchy problem (2. 3), (2. 4).

The equation $\psi = 0$ describes an entire line of force of the magnetic field. Only a portion of this line corresponds to Γ, but our analytic expression should hold on the entire line, although $g(z) = \bar{z}$ only on Γ. By pushing z beyond Γ on the line of force of which Γ is part we might hope to determine g(z) asymptotically. The problem is then defined in terms of an analytic function g(z) appearing in an integral solution of the more difficult partial differential equation (2. 3). The principal ingredient with which this general integral solution has been constructed is the Riemann function.

It is of some interest to note that in the two-dimensional cusped geometry problem, the Riemann function is identically 1 and the boundary conditions are

$$\psi = 0 \quad , \quad \frac{\partial \psi}{\partial n} = 1$$

on Γ. Moreover we are in the plane where $z^* = \bar{z}$ whence $\overline{z^*} = z$ so that

$$\psi(z, \bar{z}) = \frac{1}{2} \int_{\overline{z^*}}^{z} (1 \cdot 1) ds = \frac{1}{2} \int_{z}^{z} ds = 0$$

as required by the boundary conditions.

Now, before continuing with our analysis of the axisymmetric problem we must examine the behavior of the hypergeometric function which appears in the integrand of (2. 21).

3. HYPERGEOMETRIC FUNCTION

As is well known the hypergeometric equation

$$w(1 - w)\frac{d^2F}{dw^2} + \left\{c - (a + b + 1)w\right\}\frac{dF}{dw} - abw = 0$$

is satisfied by the hypergeometric function

$$F(a, b;c;w) = \sum_{n=o}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(n+1)} w^n$$

where $\Gamma(n+1) = n!$ is the gamma function. The hypergeometric function is absolutely convergent for $|w|<1$, and divergent for $|w|>1$[19]. When $|w| = 1$ the function is absolutely convergent if [7, 14, 16, 19]

$$\mathrm{Re}(c - a - b) > 0$$

Consider the identity [18]

$$(3.1)\quad c\left\{c - 1 - (2c - a-b - 1)w\right\} F(a, b; c; w) + (c - a)(c - b)wF(a, b; c+1; w) = c(c - 1)(1 - w)F(a, b; c - 1; w)$$

for

$$\mathrm{Re}(c - a - b) > 1.$$

The left side of this equation is absolutely convergent at $z = 1$ for $\mathrm{Re}(c - a - b) > 0$ but the right side requires $\mathrm{Re}(c - a - b) > 1$. Then all three hypergeometric functions are absolutely convergent at $w = 1$ and, by Abel's theorem [19] their limits as $w \to 1$ are their values at $w = 1$.

By letting $w \to 1$ in (3.1) we obtain the identity [19]

$$F(a, b; c; 1) = \frac{(c - a)(c - b)}{c(c - a - b)} F(a, b; c+1; 1)$$

which is iterated to give [19]

$$F(a, b; c; 1) = \lim_{m\to\infty} \prod_{n=0}^{m-1} \frac{(c-a+n)(c-b+n)}{(c+n)(c-a-b+n)} \lim_{m\to\infty} F(a, b; c+m; 1)$$

The first limit is clearly

$$\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}$$

and the second limit is shown [19] to be unity. Therefore, if $\operatorname{Re}(c - a - b - 1) > 0$ we have

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \tag{3.2}$$

and, in particular

$$F(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; 1) = \frac{4}{\pi} \tag{3.3}$$

4. THE CUSPS

In a previous section of this paper we considered an axial cross-section of the spindle as shown in figure 3. This cross-section has two axes of symmetry and consists of four mirror image curves obtainable from each other by reflection across the axes of symmetry. The longitudinal axis of symmetry, which intersects the point cusps, is the x-axis of the complex z-plane, and the other axis of symmetry is parallel to the y-axis of the z-plane. The cross-section may be displaced in the x-direction at will, but not in the y-direction. This is because the differential equation (2.1) governing the magnetic field in its exterior is invariant with respect to x-translation but not with respect to y-translation.

The curve $\Gamma_A (\Gamma_B)$, together with those portions of the axes of symmetry lying above it and to its right (left), forms a line of force $L_A (L_B)$ of the magnetic field in which the plasma is suspended. By scaling the curve Γ_A we can place the edge cusp at $y = 1$ on the y-axis. This leaves Γ_A in the first quadrant, which will prove quite helpful as the starting point of our analysis of the edge cusp. Distinct advantages in calculation at the point cusp are realized if this cusp lies at the origin and if the curve describing one arc of it lies in the first quadrant, as is the case when the point cusp of Γ_B is placed at the origin. This will be the starting point of our point cusp analysis. For convenience, in our separate analyses of the point and edge cusps we will drop the subscripts A and B from Γ_A and Γ_B because there will be no danger of confusion. When stating final results the distinction will again be noted, and the expansion for a single case will be given. We begin our separate analyses of the two cusps with the simpler of the two, the edge cusp.

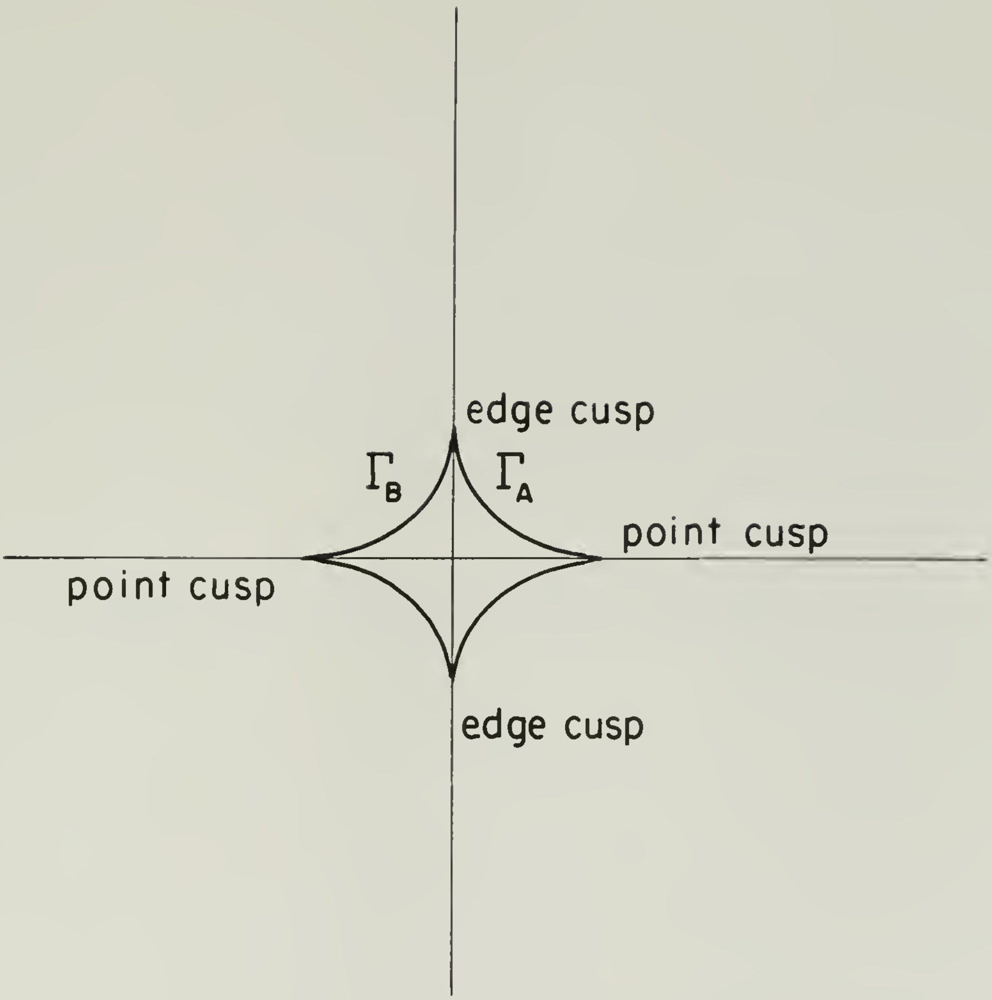


Figure 3. Axial Cross-Section of the Plasma Spindle.

5. THE EDGE CUSP

Confluence of analytic boundary conditions

Figure 4 depicts the edge cusp as it will be analyzed here. As shown in section 2, our problem is to make the integral I_0 defined by formula (2. 21) real on the line of force, L, at least in the neighborhood of the cusps, subject to the condition that the analytic function g(z) appearing in I_0 satisfy the equation $g(z) = \bar{z}$ on Γ. There is no difficulty in proving that I_0 is real on Γ itself, but the situation is quite different on the y-axis. The limits of integration of I_0 are at our disposal. We choose them to be the point of separation i and a variable point iy on the imaginary axis, so that the path of integration will lie on the positive y-axis. Then the problem becomes as follows

$$I_0 = \int_i^{iy} F\left(\frac{(iy - t)(-iy - g(t))}{(iy - g(t))(-iy - t)}\right)\sqrt{(-iy - t)(iy - g(t))g'(t)}\, dt = \text{real} \tag{5. 1}$$

on the y-axis for $y \geqslant 1$, and

$$g(z) = \bar{z} \tag{5. 2}$$

on Γ.

The only function at our disposal is the analytic function g(z). On the segment of the positive y-axis from i to iy it must be an analytic function satisfying the integral equation (5. 1) and on Γ it must satisfy $g(z) = \bar{z}$. Thus at the edge cusp z = i we have a confluence of analytic boundary conditions. Our ultimate goal in this section is to obtain the asymptotic shape of Γ in the neighborhood of the edge cusp. This is defined by the asymptotic behavior of g(z) in that region.

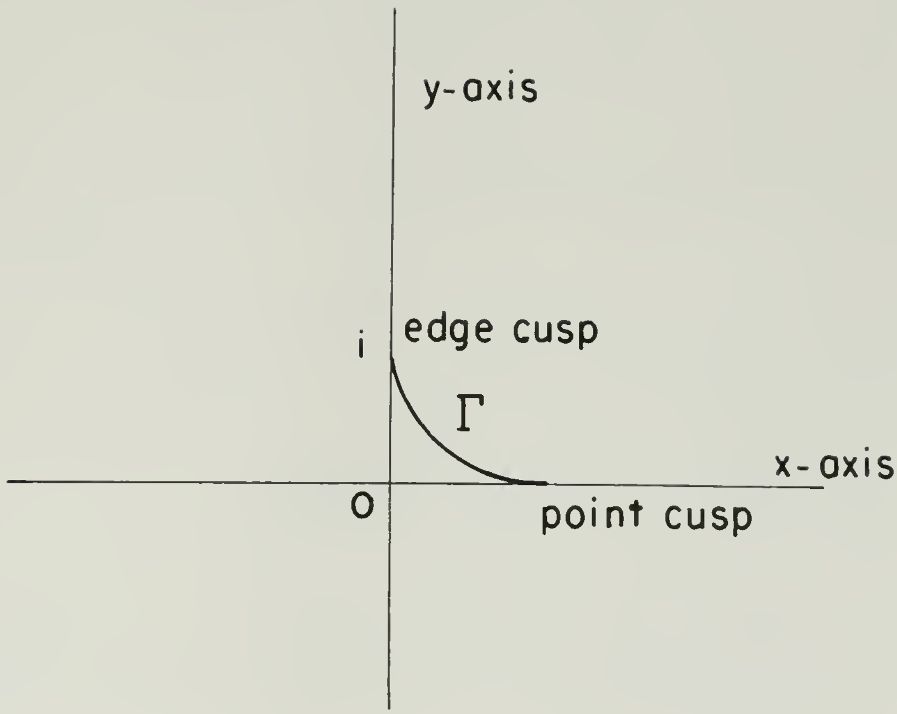


Figure 4. Edge Cusp at $z=i$.

We note that if g(t) = pure imaginary for t on the imaginary axis, then (iy - t), (-iy - g(t)), (iy - g(t)), and (-iy - t) are all pure imaginary. In the integrand this makes the argument of F real, so that F itself is real. Moreover, because g(t) is near -i for t near i, (-iy - t)(iy - g(t)) = real > 0. Of course, g' (t) is then real because it is the change in a pure imaginary with respect to a pure imaginary. If g' (t) is negative, its square root is imaginary, and thus the whole integrand of I_0 becomes imaginary. Combined with dt, which is imaginary, this makes I_0 = real. If we have g' (t) > 0, then I_0 = pure imaginary, contrary to our requirement.

On the basis of the above observations it is reasonable to attempt to prove deductively that g(t) is pure imaginary for t on the imaginary axis. If I_0 is differentiated with respect to iy and the resultant equation is solved for $\sqrt{-g'(iy)}$, we obtain

$$\text{(5.3)} \qquad \sqrt{-g'(iy)} = \frac{-\dfrac{dI_0}{dy}}{\sqrt{(2y)(y+ig(iy))}}$$

$$+ \int_i^{iy} \left\{ 2F'(w)\sqrt{(-iy-t)(iy-g(t))}\,\left[\frac{i(t-g(t))(tg(t)-y^2)}{(iy-g(t))^2(-iy-t)^2}\right]\right.$$

$$\left. + \frac{F(w)}{2}\left[\frac{2y-i(t-g(t))}{\sqrt{(-iy-t)(iy-g(t))}}\right]\right\}\left\{\sqrt{(2y)(y+ig(iy))}\right\}^{-1}\sqrt{-g'(t)}\,(-idt)$$

$$w = \frac{(iy-t)(-iy-g(t))}{(iy-g(t))(-iy-t)}$$

The terms appearing here are the same as those appearing in I_0, so that the bracketed term in the integrand is real for g(t) = pure imaginary. Moreover, this bracketed term is clearly bounded by some real constant M. Here we make use of the fact that F' (w) is real for real w. Because $\frac{dI_0}{dy}$ is known to be real, and no further information concerning it will be needed, we regard it as a known quantity. If we set $\sqrt{-g'(t)} = u(t)$ we may write (5. 3) as

$$u(iy) = H_0(iy) + \int_i^{iy} H(t, g(t), iy, u(t))(-idt)$$

$$g(t) = -i - \int_i^t [u(t)]^2 \, dt$$

The integrand H satisfies a Lipschitz condition

$$|H(t, g(t), iy, u_1(t)) - H(t, g(t), iy, u_2(t))| < M|u_1(t) - u_2(t)|$$

We propose to use this condition in connection with the method of successive approximations, to show that, for g(t) pure imaginary, a unique solution to (5. 3) exists. The uniqueness of the solution assures that g(t) is pure imaginary.

Now let us define the recurrence relation

$$u_{n+1}(iy) = H_0(iy) + \int_i^{iy} H(t, g_n(t), iy, u_n(t))(-idt)$$

$$g_{n+1}(iy) = -i - \int_i^{iy} u_n^2(t) \, dt$$

$$u_0 = 1 \quad , \quad g_0 = \text{pure imaginary}$$

Our key observation is that if u_n = real and g_n = pure imaginary, then by the above arguments u_{n+1} = real and g_{n+1} = pure imaginary, too. Thus if u_n, g_n converge to u and g, then u and g respectively must also be real and pure imaginary. Making use of the method of successive approximations, based on our Lipschitz condition, we obtain

$$|u_{n+1}(iy) - u(iy)| \leq \int_i^{iy} |H(t, g(t), iy, u_n(t)) - H(t, g(t), iy, u(t))| (-idt)$$

$$< M|u_n(t) - u(t)| \int_i^{iy} (-idt) = M|u_n(t) - u(t)|(y-1)$$

By iterating this relationship we find

$$|u_{n+1}(t) - u(t)| \leq M^{k+1}(y-1)^{k+1}|u_{n-k}(t) - u(t)|$$

$$\leq M^{n+1}(y-1)^{n+1}|1 - u(t)|$$

Because $1 - u(t)$ is bounded and $(y - 1)$ may easily be chosen so that $|y - 1| < (1/M)$ we conclude that u_n converges to u. Every u_n = real which implies that every g_n = pure imaginary and that the limit of the u_n is real. Therefore $u = \sqrt{-g'(iy)}$ = real. This means that $g'(iy)$ = real < 0, which confirms that I_0 = real if and only if g(t) is pure imaginary on the segment of the imaginary axis. The argument of the hypergeometric function in the integrand of I_0 is real and bounded arbitrarily close to zero.

Expansion of g(z)

Having proven that g(z) = pure imaginary on the y-axis, near the edge cusp, we note the obvious fact that both z + g(z) and z - g(z) are pure imaginary here. On Γ the known relationship $\bar{z} = g(z)$ tells

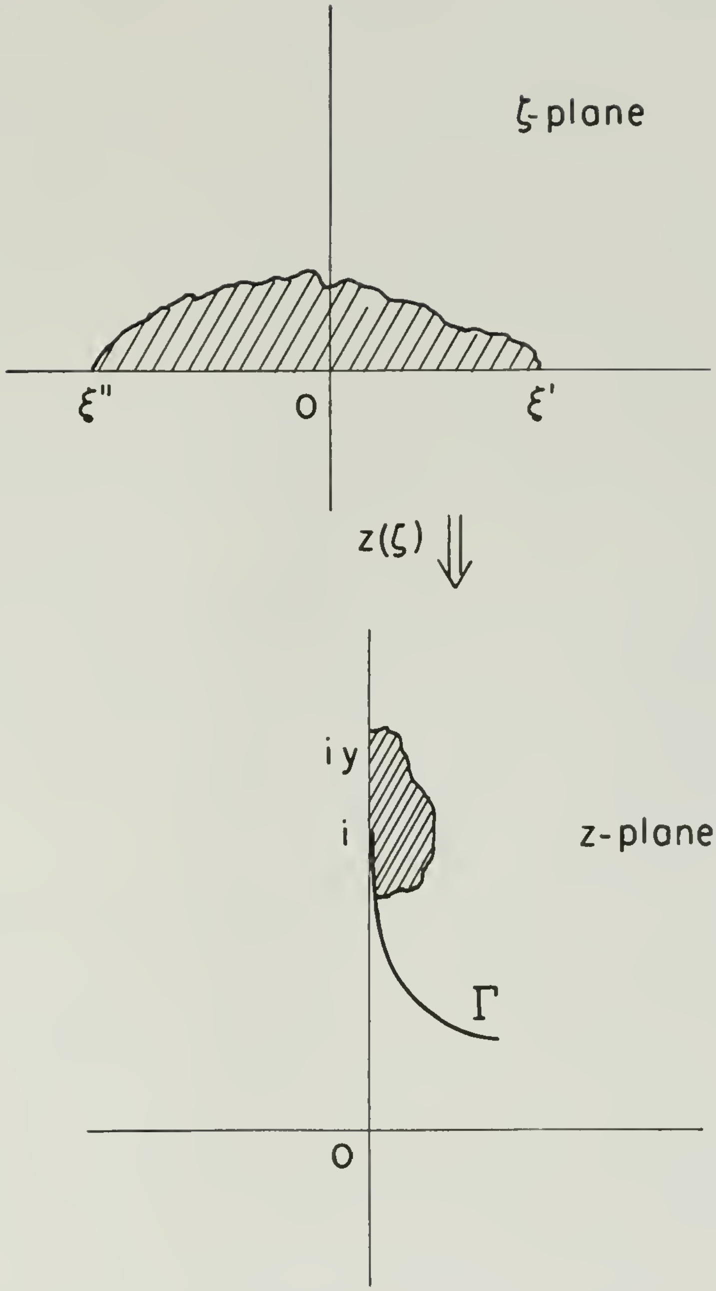


Figure 5. Mapping a Semi-Neighborhood of the Origin of the ζ-Plane Upon a Semi-Neighborhood of the Edge Cusp.

that $z + g(z) = 2x$ = real and $z - g(z) = 2iy$ = pure imaginary there. Now consider the ζ-plane where $\zeta = \xi + i\eta$. Let us map a semineighborhood $(\eta > 0)$ of the origin of the ζ-plane conformally onto a semineighborhood of i to the right of Γ and the y-axis (from i to iy) so that $\zeta = 0$ goes into $z = i$, the segment of the ξ - axis $0 \leq \xi \leq \xi'$ goes into Γ, and the segment of the ξ - axis $0 \geq \xi \geq \xi''$ goes into the segment of the y-axis $1 \leq t \leq y$. Then $z - g(z)$ is pure imaginary on the ξ - axis and $z + g(z)$ is real on the positive ξ - axis and pure imaginary on the negative ξ - axis.

Both $-i(z-g(z))$ and $\sqrt{\zeta}(z+g(z))$ are real on the real ξ - axis. Schwarz's reflection principle permits both to be reflected below the ξ - axis, whereby they become analytic in the neighborhood of $\zeta = 0$. Consequently these functions may be expanded in convergent power series

$$-i(z - g(z)) = \sum_{n=0}^{\infty} A_n \zeta^n \qquad (A_n = \text{real})$$

$$\sqrt{\zeta}(z + g(z)) = \sum_{n=0}^{\infty} B_n \zeta^n \qquad (B_n = \text{real})$$

which may be written more clearly as

$$z - g(z) = \sum_{n=0}^{\infty} iA_n \zeta^n \tag{5.4}$$

$$z + g(z) = \sum_{n=0}^{\infty} B_n \zeta^{n - \frac{1}{2}} \tag{5.5}$$

If we set

$$C_{2n-1} = (-1)^n \frac{B_n}{2} \; ; \quad C_{2n} = (-1)^n \frac{A_n}{2}$$

we may combine (5.4) and (5.5) to obtain z and g(z). Then we have

$$z = \sum_{k=-1}^{\infty} (i)^{k+1} C_k \zeta^{\frac{k}{2}}$$

$$g(z) = \sum_{k=-1}^{\infty} (-i)^{k+1} C_k \zeta^{\frac{k}{2}}$$

From the geometry of the cusp we see that $z(\zeta)$ and $z'(\zeta)$ are bounded in the neighborhood of $z = i$. In order that this remain true as $\zeta \to 0$, we must have $C_{-1} = C_{+1} = 0$. The known correspondence $z(o) = i$ tells us that $C_0 = 1$. Thus far we have not scaled the ζ-plane in relation to the z-plane. We do so now by requiring $C_2 = 1$. Then we may write

$$z = i - i\zeta + C_3 \zeta^{\frac{3}{2}} + \sum_{k=4}^{\infty} (i)^{k+1} C_k \zeta^{\frac{k}{2}}$$

$$g(z) = -i + i\zeta + C_3 \zeta^{\frac{3}{2}} + \sum_{k=4}^{\infty} (-i)^{k+1} C_k \zeta^{\frac{k}{2}}$$

For the right half of the upper cusp we must have $C_3 > 0$ in order that Γ be in the first quadrant. For the left half of the upper cusp, we require $C_3 < 0$ in order that Γ be in the second quadrant. For the lower cusp we take the conjugate of these expressions, which yield complete analytic descriptions of the edge cusp. They are convergent so that we do not have to be satisfied with asymptotic expansions.

The most important term in the above is $\zeta^{\frac{3}{2}}$, which determines the nature of the cusp. The exponent is exactly the same as is obtained in the two-dimensional case. [2]

We are, of course, concerned with g(z) as a function of z rather than of ζ. This expansion is obtained by eliminating ζ from the formulas for z and g(z), which gives

$$g(z) = -z + 2iC_3(z-i)^{\frac{3}{2}} + \sum_{n=4}^{\infty} a_n(z-i)^{\frac{n}{2}}$$

If the cusp is located at $z = z_1 = x_1 + iy_1$ instead of z=i, this becomes

$$(5.6) \qquad g(z) = -z + 2iC_3(z-z_1)^{\frac{3}{2}} + \sum_{n=4}^{\infty} a_n(z-z_1)^{\frac{n}{2}}$$

Here again the exponent $\frac{3}{2}$ shows up. As the C_n are undetermined there is no advantage in listing formulas relating the a_n and C_n.

The above expansion of g(z) substituted into $\bar{z} = g(z)$ provides an exact analytic description of the edge cusp in terms of the complex variable z. It is possible to expand this equation asymptotically in terms of the real variables x and y as follows

$$x = \frac{-C_3}{\sqrt{2}}(|y_1-y|)^{\frac{3}{2}} - \frac{3C_3}{4}(|y_1-y|)^2 + 0(|y_1-y|^{\frac{5}{2}})$$

6. THE POINT CUSP: PART I

Singular nature of the point cusp

Figure 6 depicts the point cusp as it will be analyzed. Section 4 of this paper introducing the cusp analyses describes how this figure is arrived at. Here Γ, together with the negative x-axis and the line parallel to the y-axis, extending upward from the edge cusp, constitutes a line of force of the magnetic field in which the plasma is suspended.

Our problem at the point cusp is much like that of the edge cusp. We want the integral I_0 defined by (2.21) to be real on the above line of force, subject to the condition that the analytic function $g(z)$, appearing in I_0, satisfy the equation $g(z) = \bar{z}$ on Γ. Although I_0 is automatically real on Γ, a different situation holds on the x-axis. As we are free to choose the limits of integration of I_0, we take them to be the point of separation 0 and a variable point $x < 0$, so that the path of integration will lie on the negative real x-axis. Then our requirement becomes

$$I_0 = \int_0^x F\left[\frac{(x-t)(x-g(t))}{(x-t)(x-g(t))}\right]\sqrt{(x-t)(x-g(t))g'(t)}\,dt = \text{real}$$

Now clearly

$$\frac{(x-t)\,(x-g(t))}{(x-t)\,(x-g(t))} = 1$$

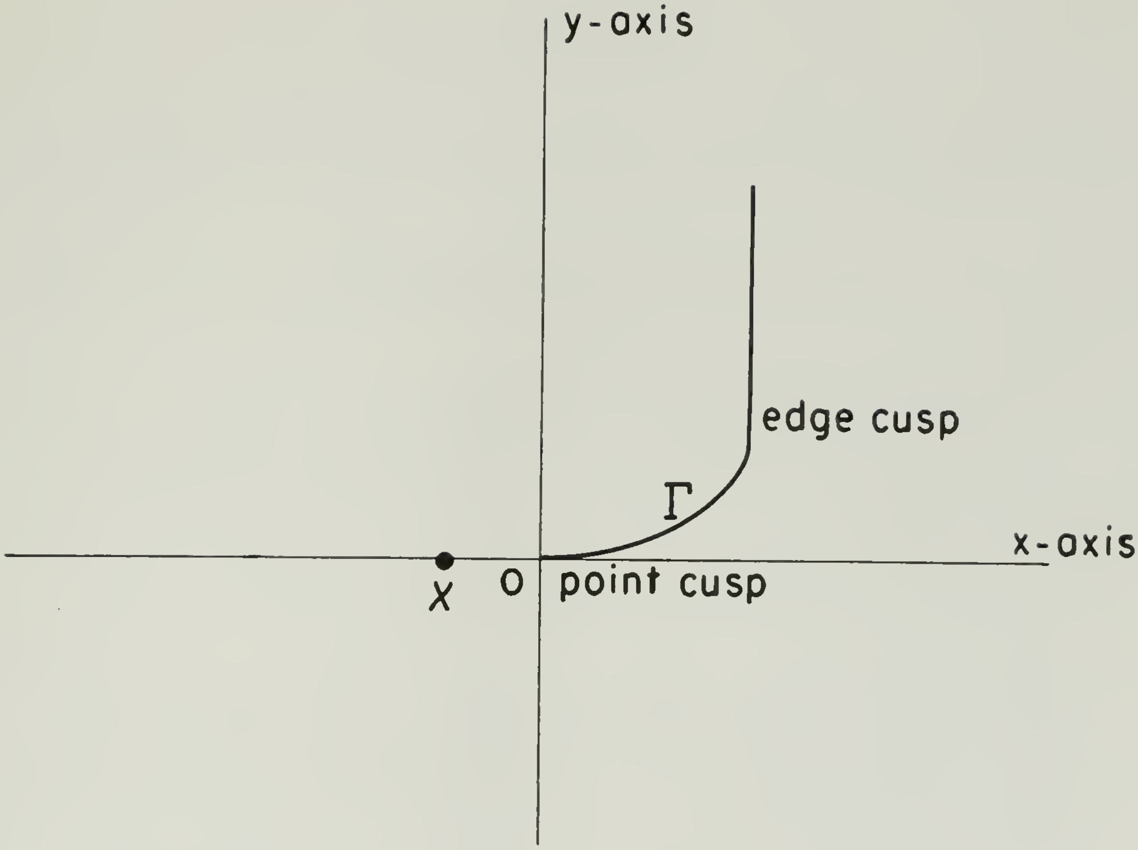


Figure 6. Point Cusp at the Origin.

and as was shown earlier in section 3 on the hypergeometric function

$$F(1) = \frac{4}{\pi}$$

because $c-a-b = 2 > 1$. Thus we set

$$I = \frac{\pi I_0}{4}$$

whereupon the mathematical statement of the problem becomes [10]

$$I = \int_0^x \sqrt{(x-t)(x-g(t))g'(t)}\, dt = \text{real} \tag{6.1}$$

on the real negative x-axis and

$$g(z) = \bar{z} \tag{6.2}$$

on Γ.

Of course, what we really want is the asymptotic shape of Γ in the neighborhood of the point cusp, which is defined by the asymptotic behavior of $g(z)$ in that region. As in the case of the edge cusp, this analytic function $g(z)$ is the only function at our disposal. It must satisfy $\bar{z} = g(z)$ on Γ and I = real on the segment $0 \geq t \geq x$. At the point cusp this segment meets Γ so that we have a confluence of two analytic boundary conditions.

The similarity of this situation to that which faced us at the edge cusp might tempt us to try the method of solution that worked there. However, differentiating I with respect to $z = x$ does not give us an integral equation for $\sqrt{\pm g'(x)}$ because our integral has

square root singularities at $t = x$ and at $g(t) = x$. Therefore the confluence of boundary conditions at the point cusp is in a certain sense singular, and the procedure employed at the edge cusp is not applicable.

Lewy's analysis

Apropos the confluence of analytic boundary conditions, the best reference is to Lewy [15], who treats a class of problems not unrelated to ours. He considers the conformal mapping of the region indicated in figure 7, but with the curve Γ known and expressed as $y = x^2 f(x)$, where $f(x)$ is a power series in x convergent for sufficiently small $|x|$. This differs markedly from our own problem, in which Γ is an unknown free boundary which we wish to determine. Lewy examines the problem of mapping the upper half ζ-plane ($\zeta = \xi + i\eta$) conformally upon a region of the z-plane so that a semi-neighborhood ($\eta > 0$) of the origin of the ζ-plane goes into a semi-neighborhood ($y > 0$) of the origin of the z-plane, $\zeta = 0$ goes into $z = 0$, a portion $0 \leq \xi \leq \xi'$ of the ξ-axis goes into Γ and a portion $\xi'' \leq \xi \leq 0$ of the ξ-axis goes into the segment $0 \geq x \geq x''$ of the x-axis. By means of difference-differential equations and integral equations, Lewy establishes that this unknown mapping $z(\zeta)$ can be expanded asymptotically, in the neighborhood of the cusp, in terms of ζ and $(\zeta \ln \zeta)$. Thus he finds

$$z(\zeta) = \sum_{m,n=0}^{\infty} A_{mn} \zeta^m (\zeta \ln \zeta)^n$$

where m and n are non-negative integers in his special example.

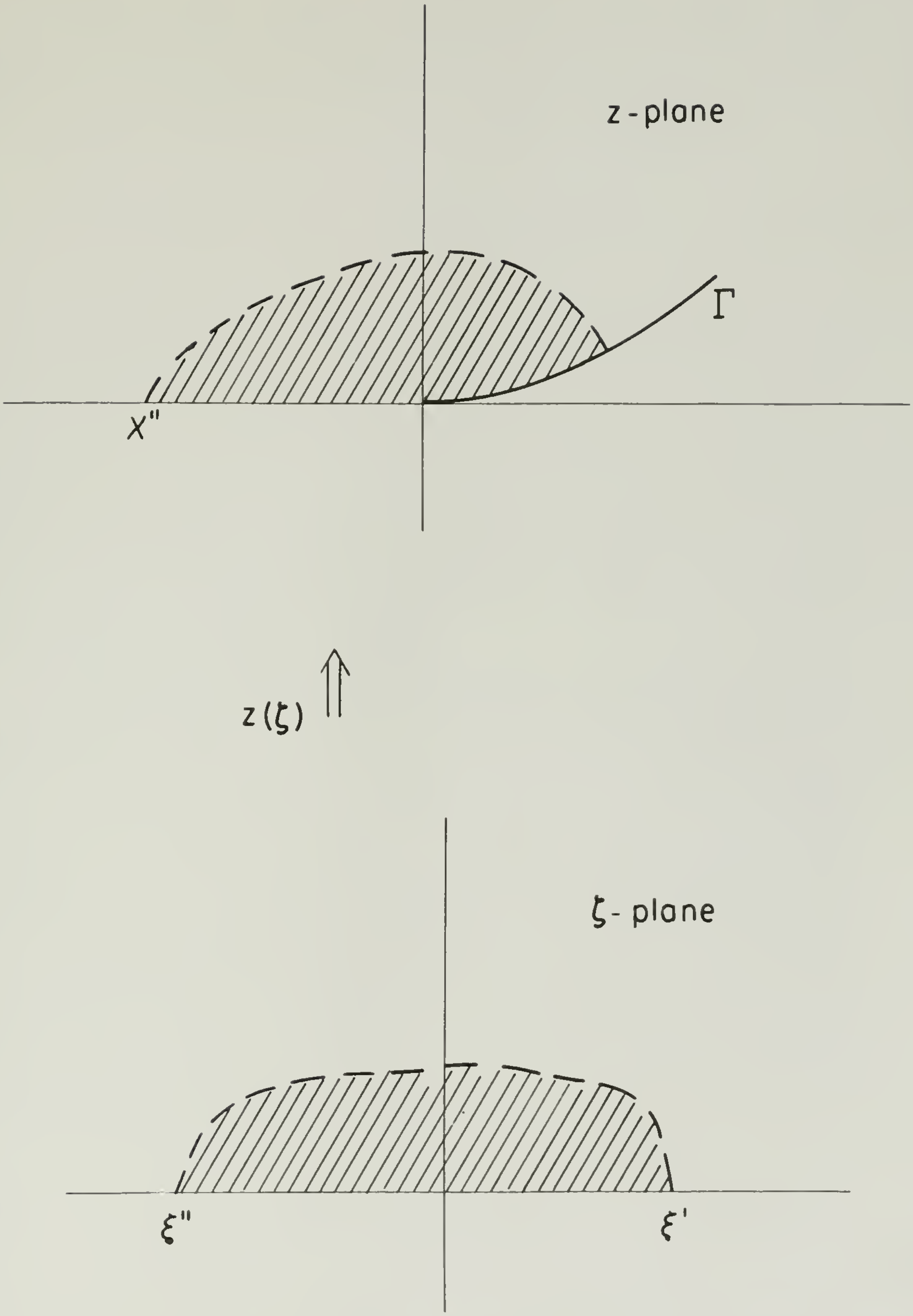


Figure 7. Mapping a Semi-Neighborhood of the Origin of the ζ-Plane Upon a Semi-Neighborhood of the Point Cusp.

Lewy's theory ought to be applicable to our problem, even though we do not know the equation of Γ. In our case the general theory suggests that g(z) might have an expansion of the form

$$g(z) = \sum_{m,n,=0}^{\infty} A_{mn} z^{\alpha_m} (z(\ln z))^{\beta_n} \qquad (\alpha_m > 0, \beta_n > 0)$$

This procedure was tried, but it failed, apparently because of the singular character of our boundary conditions at their confluence. The essential additional complicating factor not present in Lewy's analysis is the presence in I of a branch point. Even though the exact form of Lewy's asymptotic expansion of g(z) does not hold in our case, it will turn out that an expansion in increasing powers of $z^{\frac{1}{2}}$, $\ln(-\ln z)$, and $(1/\ln z)$ is valid.

We shall start by merely assuming that g(z) can be expanded in terms of the form $z^{\alpha}\hat{f}(z)$ for α = real and $\hat{f}(z)$ an analytic function. For the admissible value $\alpha = 0$ this would substitute one analytic function $\hat{f}(z)$ for another one g(z). Therefore no loss of generality is entailed. Two specific facts which are known about g(z) are that $g(0) = 0$ and that $g(z) = \bar{z}$ on Γ. The first of these facts tells us that the leading term in the expansion of g(z) is not a constant. The relation $g(z) = \bar{z}$ on Γ would not be true if the lowest order term in g(z) were z^{α} $(\alpha \neq 1)$ or $z(\ln z)^{\beta}$ $(\beta \neq 0)$. Therefore the leading term of g(z) must be z, and we may write

$$g(z) = z + az^{\alpha} f(z) \qquad (\alpha \geq 1) \tag{6.3}$$

Branch point

Before proceeding with the evaluation of α , a, and f(z) we must again turn our attention to the integral equation

$$I = \int_0^x \sqrt{(x-t)(x-g(t))g'(t)}\, dt = \text{real}$$

From (6. 3) above we know that

$$g'(t) = 1 + [\alpha a z^{\alpha - 1} f(z) + a z^{\alpha} f'(t)]$$

where the bracketed term is of higher order than 1, so that $\sqrt{g'(t)}$ may be expanded binomially. The remaining portion of the integrand is $\sqrt{(t-x)(g(t)-x)}$, which clearly vanishes when t = x and when g(t) = x. Because of the square root sign both x and $g^{-1}(x)$ are branch points of the integrand of I. For convenience we give the second of these branch points a name by defining

$$(6.4) \qquad t^* = g^{-1}(x)$$

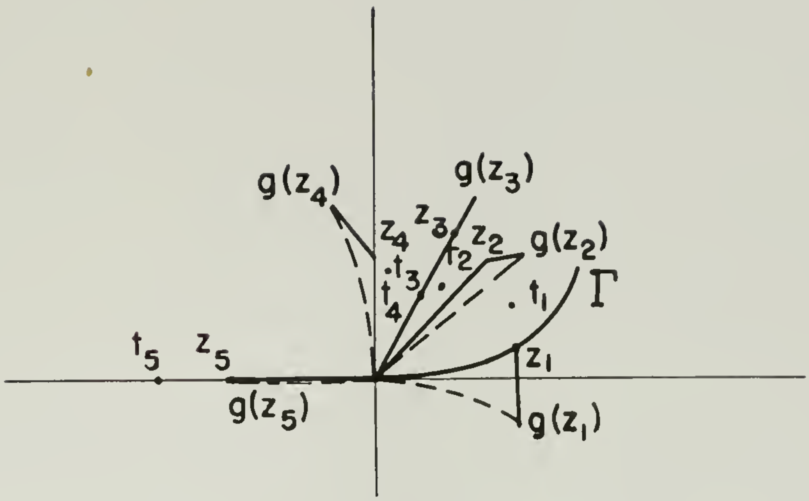
This, of course, satisfies the equation $g(t^*) = x$. We choose that branch of $\sqrt{(t-x)}$ which is real for real $t \geq x$. As this inequality holds throughout the path of integration the branch point x should not disturb us. We choose that branch of $\sqrt{(g(t)-x)}$ which is real for real $g(t) \geq x$. For real $g(t) < x$ we swing around t^* via the upper half of the z-plane so that $\sqrt{(g(t)-x)}$ becomes pure imaginary. Clearly this complicates the problem of making I real. Even if x - g(t) is not real the effect will persist. Essentially what is involved is a looping of the path of integration about the branch point. This is analyzed in the following heuristic discussion.

Suppose at first that $g(z) = z - a_0 i z^{\frac{n}{2}}$ for a_0 = real > 0 and n = integer ≥ 3. We shall consider the variation of $g(z)$ as z goes from Γ to the negative x-axis. To do so we keep $|z| = r_0$ = constant and swing z through the upper half plane to the negative x-axis. For ease of illustration we set $n = 5$. The behavior of $g(z)$ for $n = 5$ is typical of that for any other integer value of $n \geq 3$. In figure 3 we have indicated five successive positions of z, beginning with z_1 on Γ and moving through $z_2 = r_0 e^{\frac{\pi i}{4}}$, $z_3 = r_0 e^{\frac{\pi i}{3}}$, $z_4 = r_0 e^{\frac{\pi i}{2}}$, and $z_5 = r_0 e^{\pi i}$. We consider z and $z^{\frac{5}{2}}$ as vectors $r_0 e^{i\theta}$ and $r_0^{\frac{5}{2}} e^{\frac{5\theta i}{2}}$, indicated by the solid lines in figure 3. For each z_k, (k = 1, 2, 3, 4, 5) the long solid line is the path of t from $t = 0$ to $t = z_k$. The dotted line in each case represents the path of $g(t)$ from $g(0) = 0$ to $g(z_k)$. Those points marked t_k, which satisfy $g(t_k) = z_k$, are obtained approximately from

$$t_k = z_k + a_0 i z_k^{\frac{5}{2}} + 0(z_k^4)$$

We note that t_k is a branch point of $\sqrt{g(t) - z_k}$.

For $z = z_1$ the paths of t and $g(t)$ lie to the right of t_1 as t moves along the solid line to z_1 and $g(t)$ moves along the broken line to $g(z_1)$. A similar situation holds for $z = z_2$. For $z = z_3$ however, t_3 lies on the path of integration and the paths of t and $g(t)$ coincide from 0 to z_3. In order that $g(t)$ reach $g(z_3)$ precisely as t reaches z_3, it is necessary that $g(t)$ pass through z_3 before t reaches z_3. For $z = z_4$ the solid and broken lines are again separate, but both lie to the left of t_4. Thus $g(t) - z_k$ has looped around the branch point t_4. This was overlooked in the latter part of reference [10]. Thus I is not single



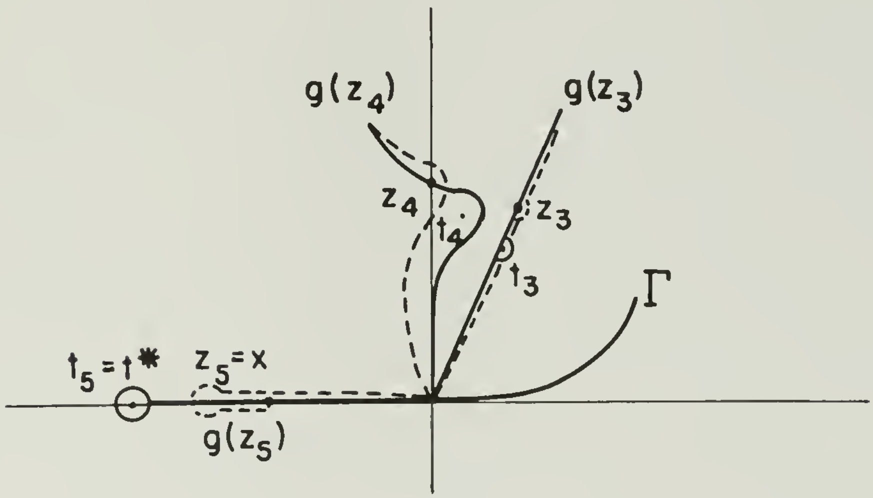


Figure 8. Integrating Around Branch Points.

valued and does not turn out to be real. For $z = z_5 = x < 0$, the situation is even worse. Here $g(x)$ doubles back from x. The branch point t_5 lies on the negative x-axis below $z_5 = x$. We note that t is really the branch point of $\sqrt{g(t) - x}$ which we previously labelled t^*.

To avoid looping about the branch point t_k we must choose the path of t so that it always passes to the right of t_k. Then $g(t)$ will keep to the right of z_k. This is illustrated in figure 8. For $z = z_3$ we cause the path of t to follow a half circle to the right of t_3 so that $g(t)$ will follow a little detour to the right of z_3. For $z = z_4$ a much greater deformation of the path of t is required to keep to the right of t_4. This keeps $g(t)$ to the right of z_4.

Our interest is primarily in $z_5 = x$. Here we lead the path of integration out to and around t^* and back under the axis to x. This causes $g(t)$ to move out to and around z_5 and back under the axis to $g(z_5)$. We would like to be free to use any path of integration in the upper half plane, especially the real negative x-axis itself. As we have just shown this cannot be done for $g(z) = z - a_0 i z^{\frac{n}{2}}$. If the branch point t^* were placed below the real negative x-axis, however, we could choose the portion of this axis from 0 to x as our path of integration for I. Then the integrand of I would not loop about t^*, as real t moves from the origin to $t = x$. Moreover, there would be no need for leading the path of integration back under the axis, or out around any particular point. Therefore we impose as a requirement of our analysis that the branch point t^* lie below the path of integration of I, which is the negative real x-axis.

In order to implement this requirement we must have some method of evaluating t^* directly. We know that $x = g(t^*)$. It will be convenient for us to obtain the inverse $t^* = h(x)$, where h, like g is analytic. We find this as follows. For z on Γ,

$$(6.5) \qquad \bar{z} = g(z) \quad ; \quad z = \bar{g}(\bar{z})$$

Here we define $\bar{g}(z) = \overline{g(\bar{z})}$, so that we have taken the conjugate of the function only. Substituting one of the relations (6.5) into the other we obtain

$$\bar{z} = g(\bar{g}(\bar{z})) \quad ; \quad z = \bar{g}(g(z))$$

These algebraic identities in z and $\bar{z}$ tell us that $\bar{g}$ is the inverse of g. They can be used to find h(x). By definition $x = g(t^*) = g(h(x))$. We know that $x = g(\bar{g}(x))$ because $g\bar{g}$ is the identity function. Therefore

$$h = \bar{g}$$

Here, of course, we substitute $h = \bar{g}$ into $t^* = h(x)$ to obtain

$$t^* = \bar{g}(x) = x + \bar{a}x^{\alpha}\bar{f}(x)$$

Now that we have an expansion of t^* we must press on to the determination of more terms of g(z). This means that we want to find a, α, and f(z). Before a and α can be properly investigated, some information about f(z) must be obtained. Let

$$(6.6) \qquad f(z) = f_1(z) + if_2(z)$$

where, for $z = x = \text{real} > 0$, both $f_1(x)$ and $f_2(x)$ are real, $f_1(x)$ is positive, and $f_1(x)$ is of lower order than $f_2(x)$ so that f(x) is real to lowest order. These assumptions may imply that a is multiplied by an

appropriate constant. If f(z) contains any logarithmic terms we choose that branch of each logarithm whose argument is 0 on the positive real x-axis. We must keep above the branch point. Therefore as z swings over the origin to the negative real axis, via the upper half plane, the imaginary part of ln z moves from zero to $+\pi$. We also assume that any power of z which is a factor of f(z) is absorbed into z^{α}. Then, the identity $z = \bar{g}(g(z))$ requires that a be pure imaginary, i. e., $a = \pm a_0 i$, for $a_0 =$ real > 0. Because Γ is in the first quadrant and $\bar{z} = g(z)$ for z on Γ, it is necessary that the minus sign be chosen for a_0, i. e.,

$$a = -a_0 i, \; a_0 > 0$$

We know that $\alpha \geq 1$. Consider the special case $\alpha = 1$. Then

$$t^* = x + a_0 i x \bar{f}(x)$$

The distance from x to t^* is

$$|t^* - x| = a_0 |x\bar{f}(x)|$$

where f(x) is not divisible by any non-zero power of x. Thus the distance from t^* to x is almost of the same order as x, so that the branch point may be extremely close to the cusp, a situation we certainly do not want. To exclude this possibility we will restrict α to

$$\alpha > 1$$

7. THE POINT CUSP: PART II

Subdividing the path of integration

Our procedure will be to expand the integrand of I in an asymptotic series, and then to integrate the lowest order terms The imaginary part of I, so obtained, is termed the error. By choosing exponents and coefficients in the expansion of $g(z)$ in such a way that the biggest term of the error in I disappears, we determine the asymptotic expansion of $g(z)$ to lowest order. Of course, for geometric reasons, t^* must be kept below the negative x-axis and we must still have $g(z) = \bar{z}$ on Γ. The expansion of the integrand of I is not easy. The term $g'(t)$ is readily expanded by the binomial theorem, but the expansion of

$$\sqrt{(g(t) - x)} = \sqrt{t - x + at^{\alpha} f(t)}$$

is a different matter entirely. Because $t-x$ is larger than $t^{\alpha} f(t)$ for small t, but smaller than $t^{\alpha} f(t)$ for t near x, a binomial expansion is out of the question. However, a Taylor series expansion about the branch point t^* would be acceptable if we retained enough terms. If only a finite number of terms are retained we will find ourselves keeping a term in $(t-t^*)$ and discarding a term in $(t-t^*)^n$ for $n = \text{finite} > 1$, although for t close enough to t^* the $(t-t^*)$ term becomes smaller than the $(t-t^*)^n$ term is near $t = 0$.

The logical resolution of this difficulty is to break the path of integration into two parts. If the point of division t_D is chosen appropriately, it becomes possible to expand $\sqrt{g(t) - x}$ binomially on the lower section of the path while expanding $g(t)$ in a Taylor series on the upper section of the path. To accomplish the binomial expansion on the lower path we need $|x - t_D| >> |t_D^{\alpha} f(t_D)|$. To accomplish the Taylor series

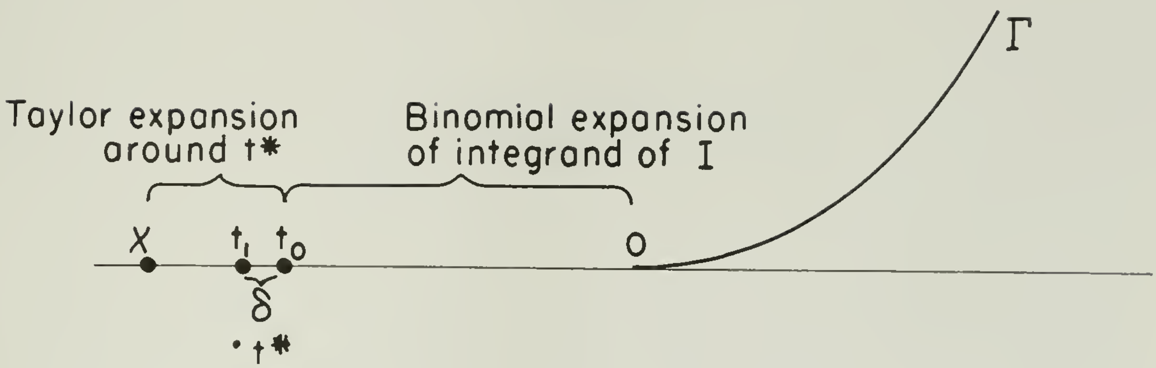


Figure 9. Subdividing the Path of Integration at t_D.

expansion on the upper path we need $|t^* - t_D| << |x|$. As noted in section 6 we confine our attention to values of α greater than 1. If $t_1 = \mathrm{Re}(t^*) = x + \mathrm{Re}(\bar{a}x^{\alpha}\bar{f}(x))$ and δ is a small positive quantity we may set $t_D = t_1 + \delta$. This will satisfy our requirements if $|x|^{\alpha} << \delta$ on the lower path and $\delta << |x|$ on the upper path. Therefore we set $\delta = |x|^{\alpha - \epsilon}$ for $0 < \epsilon < \alpha - 1$. As will become evident later, it will never be necessary to specify ϵ or δ further then this.

Now we define

$$I_1 = \int_0^{t_1 + \delta} \sqrt{(t-x)(g(t) - x)g'(t)}\, dt \tag{7.1}$$

$$I_2 = \int_{t_1 + \delta}^{x} \sqrt{(t-x)(g(t) - x)g'(t)}\, dt \tag{7.2}$$

Here I_1 and I_2 are the restrictions of the integral I to the lower and upper paths of integration, respectively. Thus we may write

$$I = I_1 + I_2 \tag{7.3}$$

By our choice of the point $t_1 + \delta$ we have insured that, in the upper interval, g(t) can be expanded in a Taylor series about the branch point t^*, as

$$g(t) = g(t^*) + g'(t^*)(t-t^*) + \frac{g''(t^*)}{2!}(t-t^*)^2 + \dots$$

$$= x + \sum_{n=1}^{\infty} \frac{g^{(n)}(t^*)}{n!}(t - t^*)^n$$

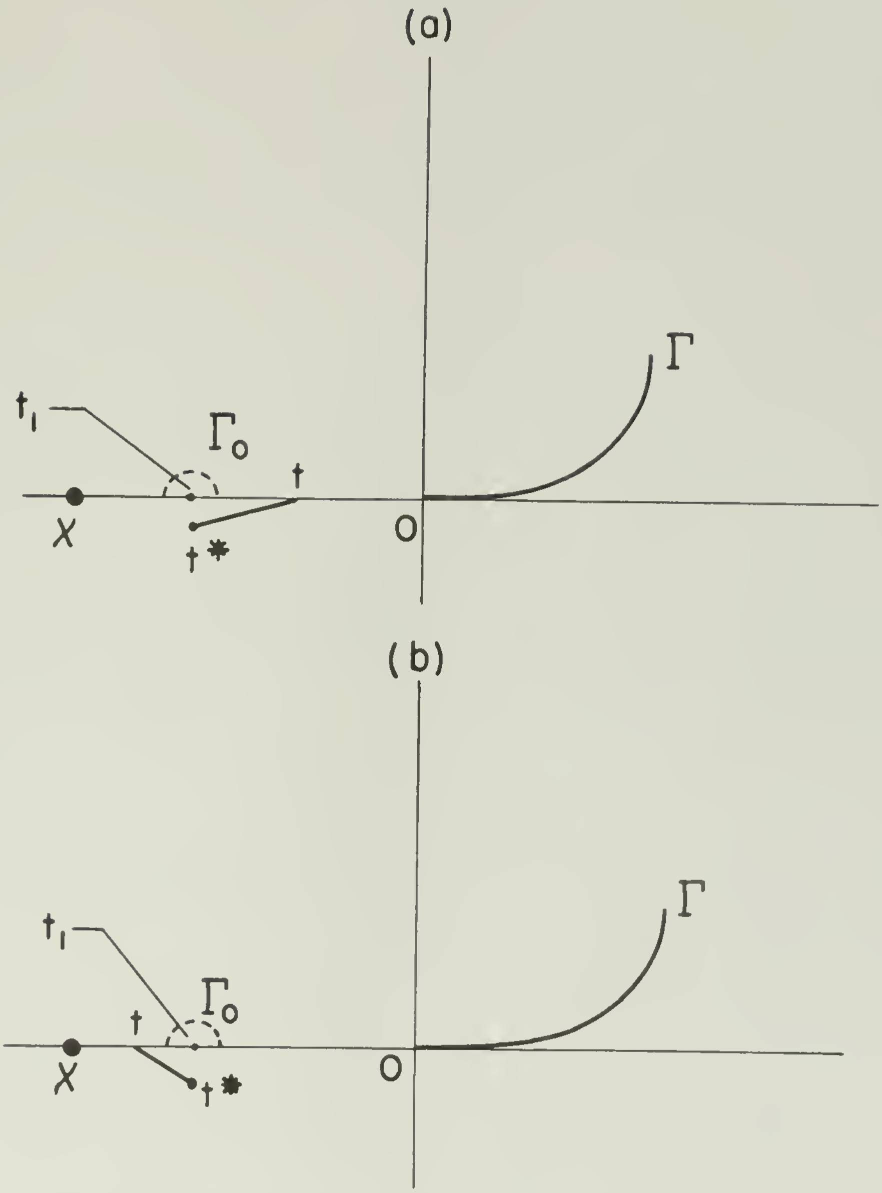


Figure 10. Avoiding the Branch Point.

From g(t) it is easy to get g' (t). Thus we have

$$g'(t) = \sum_{n=1}^{\infty} \frac{g^{(n)}(t^*)}{(n-1)!} (t - t^*)^{n-1}$$

Because g is an analytic function, all of its derivatives are defined at t^*. Moreover $|t - t^*|$ is very small so that the successive terms in $(t - t^*)^n$ shrink rapidly. If the series for g(t) and g' (t) are inserted in I_2, we obtain

$$I_2 = \int_{t_1+\delta}^{x} \sqrt{(t-x)\left(\sum_{n=1}^{\infty} \frac{g^{(n)}(t^*)}{(n-1)!} (t-t^*)^{n-1}\right)\left(\sum_{n=1}^{\infty} \frac{g^{(n)}(t^*)}{n!} (t-t^*)^{n}\right)}\ dt$$

$$= g'(t^*) \int_{t_1+\delta}^{x} \sqrt{(t-x)(t-t^*)} \left\{ 1 + \frac{3}{4} \frac{g^{(2)}(t^*)}{g^{1}(t^*)} (t-t^*) + 0((t-t^*)^2) \right\}$$

To lowest order both g' (t) and the bracketed expression in I_2 are equal to 1. Therefore, to lowest order I_2 is

$$I_2 = \int_{t_1+\delta}^{x} \sqrt{(t-x)(t-t^*)}\ dt$$

$$= \frac{1}{2} \left\{ (t - \frac{x+t^*}{2}) \sqrt{(t-t^*)(t-x)} - \frac{(x-t^*)^2}{2} \ell n \left[\frac{\sqrt{t-x} + \sqrt{t-t^*}}{2}\right] \right\}_{t_1+\delta}^{x}$$

This integral formula incorporates the term $\sqrt{(t-x)(t-t^*)}$, which branches at $t = t^*$. To avoid taking $\sqrt{(t-x)(t-t^*)}$ through the branch point, even when $t^* = t_1$, we construct a tiny semi-circle (Γ_0) about t_1, lying entirely in the upper half plane. For $0 \geq t \geq t_1$ the vector $t-t^*$ is as shown in figure (a) above, and $t-x$ is real and non-negative. Clearly the argument of $t-t^*$ lies between 0 and $\pi/2$. It is zero if $t^* = t_1 < t$ and $\pi/2$ if $Re(t) = t_1$.

For $t_1 \geq t \geq x$ the vector $t-t^*$ is as shown in figure (b) above, and $t-x$ is real and non-negative. Now it is clear that $\frac{\pi}{2} \leq \arg(t-t^*) \leq \pi$. Here $\arg(t-t^*)$ is $\pi/2$ if $Re(t) = t_1$ and π if $t^* = t_1 > t$. There, as $\sqrt{(t-x)(t-t^*)}$ swings along the negative real x-axis, from $0 \geq t > t_1$ along Γ_0 to $t_1 > t \geq x$ the argument of $t-t^*$ increases by an angle θ for which $0 \leq \theta \leq \pi$. Throughout this range $\arg(t-x) = 0$. Then $\arg(\sqrt{(t-x)(t-t^*)}\,)$ increases by $\frac{\theta}{2}$, with $0 \leq \frac{\theta}{2} \leq \frac{\pi}{2}$. The equal sign holds only if $t^* = t_1$. In the light of this discussion we evaluate (7. 4) at the limits of integration to obtain

$$I = \text{real} + \frac{ia_0 e^{-\alpha\pi i}}{2} |x|^{2\alpha-\epsilon} f_0(x) - \frac{a_0^{\,2}|x|^{2\alpha}}{8} \left\{ \overline{f(x)}[\epsilon\, |\ell n(x)| + \frac{1}{2} - (\alpha + \frac{3}{2})\pi i|_{\text{mod } 2\pi i}] - f_0^{\,2}(x)\, e^{-2\alpha\pi i} \right\} + 0(|x|^{2\alpha+\epsilon} |f(x)|^3)$$

Here, for real $x < 0$, we have made use of the convenient shorthand notation

$$f_0(x) = \overline{f_1(x)} + i\overline{f_2(x)} = \overline{f(x)}$$

where the functions $f_1(x)$ and $f_2(x)$ are real for $x > 0$, but complex for $x < 0$, as in (6. 6).

The lower portion of the path of integration

Next we shall turn our attention to I_1. By our choice of the point $t_1 + \delta$ we have insured that, in the lower interval, the integrand of I_1 can be expanded binomially. First we have

$$\sqrt{(t-x)(g(t)-x)} = (t-x)\left\{1 + \frac{at^{\alpha} f(t)}{2(t-x)} - \frac{a^2 t^{2\alpha} f(t)^2}{8(t-x)^2} + 0\left(\frac{t^{3\alpha} f(t)^3}{(t-x)^2}\right)\right\}$$

$$= (t-x) + \frac{at^{\alpha} f(t)}{2} - \frac{a^2 t^{2\alpha} f(t)^2}{8(t-x)} + 0\left(\frac{t^{3\alpha} f(t)^3}{(t-x)^2}\right)$$

The other term in the integrand is also expanded binomially to give

$$\sqrt{g'(t)} = 1 + \frac{a}{2}\frac{d}{dt}(t^{\alpha} f(t)) - \frac{a^2}{8}\left[\frac{d}{dt} t^{\alpha} f(t)\right]^2 + 0\left(t^{3(\alpha-1)} f(t)^3\right)$$

These two expansions are then substituted into I_1 to obtain

$$I_1 = \text{real} + \frac{a}{2}\int_0^{t_1+\delta} \left[(t-x)\frac{d}{dt}(t^{\alpha} f(t)) + t^{\alpha} f(t)\right] dt - \frac{a^2}{8}\int_0^{t_1+\delta} \frac{t^{2\alpha} f(t)^2}{(t-x)}\, dt$$

$$+ \frac{a^2}{8}\int_0^{t_1+\delta} \frac{d}{dt}(t^{2\alpha} f(t)^2)dt - \frac{(\alpha a)^2}{8}\int_0^{t_1+\delta} (t-x)t^{2\alpha-2} f(t)^2 dt$$

$$- \frac{a^2\alpha}{4}\int_0^{t_1+\delta} (t-x)t^{2\alpha-1} f(t)f'(t)\, dt + 0\left(\int_0^{t_1+\delta} t^{2\alpha}(t-x)(f'(t))^2\, dt\right)$$

Before this expansion of I_1 can be evaluated we must impose the conditions that $tf'(t)$ and $tf''(t)$ are of higher order than $f(t)$ and $f'(t)$ respectively, so that

$$\lim_{t \to 0} \frac{tf'(t)}{f(t)} = 0 \tag{7.5}$$

$$\lim_{t \to 0} \frac{tf''(t)}{f'(t)} = 0 \tag{7.6}$$

The first of these relationships is a reasonable assumption because we have already specified that f(t) contains no non-zero power of t. Therefore f(t) varies more slowly than t. Non-existence of the limit

$$\lim_{t \to 0} \frac{tf'(t)}{f(t)}$$

would imply that f(t) is highly oscillatory, which would be unnatural for the smooth curve Γ we wish to describe with $g(z) = z + az^{\alpha} f(z)$. From Lewy's work we suspect that f(t) may begin with some power of $\ell n\, t$. We will not assume this, because we wish to deduce it. The relationships (7.5) and (7.6) are, however, consistent with the possibility that the leading term of f(t) is logarithmic.

The last assumption (7.6), regarding $f''(t)$, can be dispensed with because application of (7.5) to I yields the leading term of f(t), which is enough to determine that (7.6) holds. However, we will assume (7.6) for the time being in order to expand I to several terms. Initially, only those terms dependent on (7.5) will be utilized. After the first term of f(t) is found and (7.6) substantiated we will use the terms of I dependent on (7.6) to obtain higher order terms of f(t).

With (7.5) at our disposal, it is clear that the remainder term in I_1 is smaller by order of magnitude than any of the integrals preceding it. Moreover, we can estimate the magnitude of this term. Because of (7.6),

$(t^{\alpha} f'(t))^2$ is an increasing function of t which takes its largest values in the neighborhood of the upper limit of integration, so that we may write,

$$\int_0^{t_1+\delta} t^{2\alpha}(t-x)(f'(t))^2\,dt = 0(|x|^{2\alpha+2}|f'(x)|^2)$$

The integrands of the first and third integrals of I_1 are total derivatives. Integration by parts is used for the fourth and fifth integrals to give

$$(7.7) \qquad \int_0^{t_1+\delta} (t-x)t^{2\alpha-2}(f(t))^2\,dt = \left(\frac{t^{2\alpha}}{2\alpha} - \frac{xt^{2\alpha-1}}{2\alpha-1}\right)(f(t))^2 \Bigg|_0^{t_1+\delta}$$

$$- \frac{1}{\alpha}\left(\frac{t^{2\alpha+1}}{2\alpha+1} - \frac{xt^{2\alpha}}{2\alpha}\right) f(t)f'(t) \Bigg|_0^{t_1+\delta}$$

$$+ \frac{1}{\alpha}\int_0^{t_1+\delta} \left(\frac{t^{2\alpha+1}}{2\alpha+1} - \frac{xt^{2\alpha}}{2\alpha}\right) ((f'(t))^2 + f(t)f''(t))\,dt$$

$$(7.8) \qquad \int_0^{t_1+\delta} (t-x)t^{2\alpha-1}f(t)f'(t)\,dt = \left(\frac{t^{2\alpha+1}}{2\alpha+1} - \frac{xt^{2\alpha}}{2\alpha}\right) f(t)f'(t) \Bigg|_0^{t_1+\delta}$$

$$- \int_0^{t_1+\delta} \left(\frac{t^{2\alpha+1}}{2\alpha+1} - \frac{xt^{2\alpha}}{2\alpha}\right) (f'(t)^2 + f(t)f''(t))\,dt$$

Because of (7. 5) and (7. 6) the integrals on the far right of (7. 7) and (7. 8) are of the order of the remainder term in I_1.

With the aid of the substitution $t = x(1 - \tau)$ the second integral of I_1 becomes

$$(7.9) \qquad \int_0^{t_1+\delta} \frac{t^{2\alpha} f(t)^2}{t-x}\, dt = -x^{2\alpha} \int_\omega^1 (1-\tau)^{2\alpha} f(x(1-\tau))^2 \frac{d\tau}{\tau}$$

$$= -x^{2\alpha} \int_\omega^1 f(x(1-\tau))^2 \frac{d\tau}{\tau} - x^{2\alpha} \int_\omega^1 f(x(1-\tau))^2 \sum_{m=1}^{N} \binom{2\alpha}{m} (-\tau)^m d\tau$$

where N equals 2α or ∞ as 2α is or is not an integer, and $\omega = 1 - \left(\frac{t_1+\delta}{x}\right) = |x|^{\alpha-1-\epsilon} - 0(|x|^{\alpha-1}|f(x)|)$. Then the next to last integral in (7. 9) is integrated by parts to obtain

$$(7.10) \quad -x^{2\alpha} \int_\omega^1 f(x(1-\tau))^2 \frac{d\tau}{\tau} = -x^{2\alpha} (\ell n\tau) f(x(1-\tau))^2 \Big|_\omega^1 + 2x^{2\alpha+1} \int_\omega^1 (\ell n\tau) f(x(1-\tau)) f'(x(1-\tau))\, d\tau$$

$$= x^{2\alpha} \ell n |x|^{\alpha-1-\epsilon} f(x)^2 - 2x^{2\alpha+1} (\tau \ell n\tau - \tau + 1) f(x(1-\tau)) f'(x(1-\tau)) \Big|_\omega^1$$

$$+ 2x^{2\alpha+2} \int_\omega^1 (\tau \ell n\tau - \tau + 1)(f'(x(1-\tau))^2 + f(x(1-\tau)) f''(x(1-\tau)))\, d\tau +$$

$$+ \ 0(|x|^{3\alpha-\epsilon}|\ell n|x|| \ |f(x)||f'(x)|)$$

$$= (\alpha-1-\epsilon)x^{2\alpha}\ell n|x|(f(x))^2 + 2x^{2\alpha+1}f(x)f'(x)$$

$$+ \ 2x^{2\alpha+2}\int_{\omega}^{1} (\tau\ell n\tau-\tau+1)(f'(x(1-\tau))^2 + f(x(1-\tau))f''(x(1-\tau)))\,d\tau$$

$$+ \ 0(|x|^{3\alpha-\epsilon}|\ell n|x|| \ |f(x)||f'(x)|)$$

The last integral on the right in (7.10) is of the order of the remainder term in I_1. In each integration we have deliberately chosen a constant of integration which causes the partial integral to vanish at the upper limit of integration. This was done to avoid such terms as f(0) and f' (0) which might be infinite, as for instance if the leading term of f(t) should turn out to be a logarithm raised to a positive power.

The second integral of (7.9) is handled similarly to the first one.

$$(7.11) \qquad - \ x^{2\alpha}\int_{\omega}^{1} f(x(1-\tau))^2 \left(\sum_{m=1}^{N} \binom{2\alpha}{m}(-\tau)^m\right) d\tau$$

$$= \ - \ x^{2\alpha} \sum_{m=1}^{N} \binom{2\alpha}{m}(-1)^m \left\{ \frac{f(x)^2}{m+1} - \frac{2x}{m+2} f(x)f'(x) \right.$$

$$\left. + \ \frac{2x^2}{m+1}\int_{\omega}^{1} \left(\frac{\tau^{m+2}}{m+2} - \tau + \frac{m+1}{m+2}\right)(f'(x(1-\tau))^2 + f(x(1-\tau))f''(x(1-\tau)))\,d\tau \right\} +$$

$$+ \quad 0(|x|^{3\alpha-\epsilon}|f(x)||f'(x)|) + 0(|x|^{4\alpha-2-2\epsilon}|f(x)|^2)$$

The last integral on the right of (7.11) is of the order of the remainder term in I_1. Again we have chosen the constant of integration to avoid such terms as f(0) and f'(0). Combining the results (7.7), (7.8), (7.10), and (7.11), we have

$$I_1 = \text{real} + \frac{a}{2}(t-x)t^{\alpha} f(t)\Bigg|_0^{t_1+\delta} - \frac{a^2}{8}\Bigg\{x^{2\alpha} f(x)^2 \ell n|x|^{\alpha-1-\epsilon} - x^{2\alpha} f(x)^2 \sum_{m=1}^{N} \binom{2\alpha}{m} \frac{(-1)^m}{m+1}$$

$$+ \; 2\sum_{m=1}^{N} \binom{2\alpha}{m}(-1)^m \left(\frac{m+3}{m+2}\right) f(x)f'(x)\Bigg\} + \frac{a^2}{8} t^{2\alpha} f(t)^2 \Bigg|_0^{t_1+\delta}$$

$$- \; \frac{a^2}{8}\Bigg\{\alpha^2 \left(\frac{t^{2\alpha}}{2\alpha} - \frac{xt^{2\alpha-1}}{2\alpha-1}\right) f(t)^2$$

$$+ \; [\frac{\alpha t^{2\alpha+1}}{2\alpha+1} - xt^{2\alpha}(1 - \frac{\alpha}{2\alpha-1})]\, f'(t)f(t)\Bigg\}_0^{t_1+\delta}$$

$$+ \; 0(|x|^{2\alpha+2}|f'(x)|^2) + 0(|x|^{4\alpha-2-2\epsilon}|f(x)|^2)$$

Most of these terms vanish at t = 0. When the above expression is evaluated at the limits of integration it becomes

$$I_1 = \text{real} - \frac{a_0 i}{2} |x|^{2\alpha-\epsilon} f(x) e^{\alpha\pi i} - \frac{a_0^2 |x|^{2\alpha} e^{2\alpha\pi i}}{8} \Big\{ (\alpha-1-\epsilon) \big| \ell n |x| \big| f(x)^2$$

$$+ \sum_{m=1}^{N} \binom{2\alpha}{m} \frac{(-1)^m}{(m+1)} f(x)^2 + 2 \sum_{m=1}^{N} \binom{2\alpha}{m} (-1)^m \frac{m+3}{m+2} |x| f(x) f'(x) \Big\}$$

$$- \frac{a_0^2 |x|^{2\alpha}}{8} f(x)^2 e^{2\alpha\pi i} - \frac{a_0^2 |x|^{2\alpha} e^{2\alpha\pi i}}{8} \left\{ \frac{\alpha f(x)^2}{2(2\alpha-1)} + \frac{|x| f'(x) f(x)}{4\alpha^2 - 1} \right\}$$

$$+ 0(|x|^{4\alpha-2-2\epsilon} |f(x)|^2) + 0(|x|^{2\alpha+2} |f'(x)|^2)$$

<u>Exponents of the lowest order terms in g(x) chosen to reduce the error in I</u>

To calculate I we need only add I_1 and I_2. Before doing so, however, we recall that $f(x) = f_1(x) + if_2(x)$ where $f_1(x)$ and $f_2(x)$ are real for x real and positive, and complex otherwise. Then we may write

$$I_2 = \text{real} + \frac{ia_0 |x|^{2\alpha-\epsilon}}{2} (\overline{f_1(x)} + \overline{if_2(x)}) e^{-\alpha\pi i} - \frac{a_0^2 |x|^{2\alpha} \overline{f}(x)^2 e^{2\alpha\pi i}}{8} (\epsilon \big| \ell n |x| \big| + 0(1))$$

$$I_1 = \text{real} - \frac{ia_0 |x|^{2\alpha-\epsilon}}{2} (f_1(x) + if_2(x)) e^{\alpha\pi i} - \frac{a_0^2 |x|^{2\alpha} f(x)^2 e^{2\alpha\pi i}}{8} ((\alpha-1-\epsilon) \big| \ell n |x| \big|$$

$$+0(1))$$

Adding, we obtain

$$I = I_1 + I_2 = \text{real} + ia_0 |x|^{2\alpha - \epsilon} \operatorname{Im}(f_2(x) e^{\alpha\pi i})$$

$$- \frac{a_0^2 |x|^{2\alpha} e^{2\alpha\pi i}}{8} \left\{ (f_1(x)^2 - f_2(x)^2)(\alpha - 1) \left| \ln |x| \right| + 2i f_2(x) f_1(x) (\alpha - 1 - 2\epsilon) \left| \ln |x| \right| + 0(1) \right\}$$

The constant ϵ arises from the definition of $\delta = |x|^{\alpha - \epsilon}$. This, in turn, occurs in $t_1 + \delta$, which is the arbitrarily chosen point at which the path of integration of I is broken to define I_1 and I_2. Of course I must be independent of ϵ and δ. This means that the terms including $|x|^{2\alpha - \epsilon}$ and $(\alpha - 1 - 2\epsilon)$ must vanish from I. As this only happens for $f_2(x) = 0$, we put

$$f_2(x) = 0(|x|^{\lambda}), \qquad \lambda > 0$$

Then I becomes, for $\alpha > 1$,

$$I = \text{Real} - \frac{a_0^2 |x|^{2\alpha} e^{2\alpha\pi i}}{8} \left\{ f_1(x)^2 (\alpha - 1) \left| \ln |x| \right| + 0(1) \right\}$$

Because $f_1(x)$ is defined to be real for real $x > 0$, and to have no non-zero power of x as a factor, $f_1^2(x)$ must be real to lowest order for $x = \text{real} < 0$. Then the largest error term in I vanishes for $2\alpha = \text{integer}$. For $\alpha > 1$ the principal solution would be $\alpha = 3/2$. Solutions for $\alpha = 2, 5/2, 3, 7/2$, etc. are higher order solutions that may be expected to occur only in degenerate or special examples. These higher values of α correspond exactly to the exponents of the higher order terms in the expansion of g(z) at the edge

cusp. If $C_3 = 0$ at the edge cusp, these higher order terms become solutions. We will confine our attention to the lowest exponent $\alpha = 3/2$.

Because $f_1(x)$ is complex it is convenient to our subsequent analysis to define, for x = real < 0,

$$f_{11}(x) = \mathrm{Re}\, f_1(x) \quad ; \quad f_{12}(x) = \mathrm{Im}\, f_1(x)$$

Then I becomes

$$I = \text{Real} + \frac{ia_0^2|x|^3}{8}\left\{f_{11}f_{12}(|\ell n|x|| - 2\ell n(f_{11})) - \pi(f_{11}^2 - f_{12}^2) + 0(f_{11}f_{12}) - \frac{5}{8}|x|(f_{11}f_{12}' + f_{12}f_{11}')\right\} + 0(|x|^5(f'(x))^2)$$

To make the largest error term vanish, we must have, to lowest order,

$$f_{11}(x) = \frac{|\ell n|x||}{\pi} f_{12}(x)$$

which implies

$$f_1(x) = \frac{-f_{12}(x)}{\pi}(\ell n\, x - 2\pi i)$$

For $x > 0$, $f_1(x)$ is real so that this equation becomes

$$f_{12}(x) = \frac{-\pi f_1(x)}{(\ell n\, x - 2\pi i)} = \frac{-\pi f_1(x)}{\sqrt{(\ell n\, x)^2 + (2\pi)^2}}\, e^{-i\tan^{-1}(\frac{2\pi}{\ell n x})}$$

which will hold if $f_{12}(x)$ contains the factor $(\ell n\, x - \pi i)^{-2}$. Therefore, for $x < 0$, we set, to lowest order

$$f_{12}(x) = \frac{\pi}{|\ell n|x||^2}$$

we obtain

$$f_1(x) = -\frac{(\ell n\, x - 2\pi i)}{(\ell n\, x - \pi i)} = \frac{(\ell n\, x - 2\pi i)}{\ell n\, x\,(\ell n\, x - 2\pi i) - \pi^2}$$

$$= -\frac{1}{\ell n\, x} - \frac{\pi^2}{(\ell n\, x)^3} - \frac{2\pi^3 i}{(\ell n\, x)^4} + 0\left(\frac{1}{(\ell n\, x)^5}\right)$$

Now we have determined the leading term of the asymptotic expansion of $g(z) - z$ at the point cusp, namely

$$\frac{a_0 i z^{3/2}}{(\ell n\, z)}$$

This term actually determines the shape of the point cusp. If, however, we should want higher order terms in the expansion, we set

$$f_{11}(x) = \frac{1}{|\ell n|x||} + p_1(x)$$

$$f_{12}(x) = \frac{\pi}{|\ell n|x||^2} + p_2(x)$$

where $p_1(x)$ and $p_2(x)$ are real and of higher order than $|\ell n|x||^{-1}$ and $|\ell n|x||^{-2}$ respectively. Substituting $f_1(x) = f_{11}(x) + if_{12}(x)$ into I we obtain the following:

$$I = \text{real} + \frac{a_0^2|x|^3\pi i}{8|\ell n|x||^3}\left\{-2\,\ell n|\ell n|x|| - p_1(x)|\ell n|x||^2 + \frac{p_2(x)|\ell n|x||^3}{\pi} + 0\left(\frac{1}{|\ell n|x||}\right)\right\}$$

In general we will find the largest error term in I to be

$$\frac{i\pi a_0^2|x|^3}{8|\ell n|x||^n}\left\{C_n - v_1(x)|\ell n|x||^{n-1} + \frac{v_2(x)|\ell n|x||^n}{\pi}\right\}$$

where n = integer, C_n contains some power of $\ell n|\ell n|x||$ and $v_1(x)$ and $v_2(x)$ are real functions. This error term vanishes for

$$v_1(x) = \frac{-D}{|\ell n|x||^{n-1}} \quad , \quad v_2(x) = \frac{-(C_n + D)}{|\ell n|x||^{n}}$$

where D is a constant. We determine D from

$$v_1(x) + iv_2(x) = \text{real function} - iG(x)$$

Here $iG(x)$ is the lowest order imaginary term produced by previously calculated terms of $f_1(x)$. Then we increase n by 1 and repeat the procedure.

To get all terms of the asymptotic expansion, we might attain a certain elegance by assuming the "Ansatz"

$$g(z) = z + \sum_{n=3}^{\infty} z^{\frac{n}{2}} \sum_{m=1}^{\infty} \frac{1}{(\ell n\, z)^m} \sum_{\ell=0}^{m-1} A_{\ell mn} (\ell n(-\ell n\, z))^{\ell}$$

where the series in $(\ell n(-\ell n\, z))^{\ell}/(\ell n\, z)^m$ ought to be convergent and the series in $z^{\frac{n}{2}}$ is asymptotic. The justification for the "Ansatz" is that we have in fact carried out the iterative scheme described above, so that we know the "Ansatz" holds for n = 3. Because of the other possible values of α the remaining terms in the "Ansatz" are plausible. Whether or not this "Ansatz" holds rigorously is an unsolved problem which will not be attacked here. In any case we find

$$g(z) = z - a_0 i z^{\frac{3}{2}} \left\{ \frac{-1}{\ell n\, z} - \frac{2\ell n(-\ell n\, z)}{(\ell n\, z)^2} + \frac{\left(-\frac{9}{4} + 2\ell n\left(\frac{a_0}{4}\right)\right)}{(\ell n\, z)^2} + 0\left(\frac{(\ell n\,(-\ell n\, z))^2}{(\ell n\, z)^3}\right) \right\}$$

$$I = \text{real} + \frac{a_0^2 |x|^3 \pi i}{8|\ell n|x||^4} \left\{ 16\,(\ell n|\ell n|x||)^2 - v_1(x)|\ell n|x||^3 + \frac{v_2(x)|\ell n|x||^4}{\pi} + 0(\ell n|\ell n|x||) \right\}$$

Consideration of g(z) as determined above tells us that we have not obtained as strong a result as an asymptotic series in which the terms contain successively higher powers of z. Nevertheless we do have $z^{3/2}$ multiplied by the convergent series

$$\sum_{m=1}^{\infty} \frac{1}{(\ell n\, z)^m} \sum_{\ell=0}^{m-1} A_{\ell m 3} (\ell n(-\ell n\, z))^{\ell}$$

and if our Ansatz holds, the expansion of g(z) does contain successively higher powers of $z^{\frac{1}{2}}$, each multiplied by a convergent series in $(\ell n(-\ell n\, z))^{\ell}/(\ell n\, z)^m$. Therefore, what we have is an acceptable asymptotic description of g(z). The curve Γ_B of figure 3 is given by $\bar{z} = g(z)$. By reflecting across the axes of symmetry of this figure the other curves are

obtained. To make these expressions compatible with the results of the edge cusp analysis we move the point cusp to some arbitrary point $z_0 < 0$ on the real negative axis. Then we have

$$(7.12) \quad g(z) = z - a_0 i(z-z_0)^{3/2} \left\{ \frac{-1}{\ell n(z-z_0)} - \frac{2\ell n(-\ell n(z-z_0))}{(\ell n(z-z_0))^2} + \frac{\left(\frac{-9}{4} + 2\ell n(\frac{a_0}{4})\right)}{(\ell n(z-z_0))^2} + 0\left(\frac{(\ell n(-\ell n(z-z_0)))^2}{(\ell n(z-z_0))^3}\right) \right\}$$

As in the case of the edge cusp, this complex equation can be expanded directly in terms of the real variables x and y to give

$$y = \frac{a_0(x-x_0)^{3/2}}{2} \left\{ \frac{-1}{\ell n(x-x_0)} - \frac{2\ell n(-\ell n(x-x_0))}{(\ell n(x-x_0))^2} + \frac{\frac{-9}{4} + 2\ell n(\frac{a_0}{4})}{(\ell n(x-x_0))^2} + 0\left(\frac{(\ell n(-\ell n(x-x_0)))^2}{(\ell n(x-x_0))^3}\right) \right\}$$

8. SUMMARY OF RESULTS

1. Proof of Riemann's representation theorem in a four dimensional space.

In the four dimensional space of two independent complex variables (z, z^*) consider the equation

$$\psi_{zz^*} + \frac{1}{2(z - z^*)} (\psi_z - \psi_{z^*}) = 0 \tag{2.3}$$

with the initial values

$$\psi = 0 \quad ; \quad \frac{2i}{(z - z^*)} \frac{\partial\psi}{\partial n} = 1 \tag{2.4}$$

given on an arbitrary noncharacteristic curve Γ. The Riemann function

$$R(z, z^*; t, t^*) = \frac{\sqrt{(z - t^*)(t - z^*)}}{(t - t^*)} F\left(\frac{(z - t)(z^* - t^*)}{(z - t^*)(z^* - t)}\right)$$

satisfies (2. 3) as a function of (z, z^*) and the adjoint of (2. 3) as a function of (t, t^*). It also satisfies

$$R_z = - \frac{1}{2(z - z^*)} R$$

on the characteristic plane $z^* =$ constant, and

$$R_{z^*} = \frac{1}{2(z - z^*)} R$$

on the characteristic plane $z =$ constant, and

$$R(z, z^*; z, z^*) = 1$$

In section 2 of this dissertation it has been proved that the solution to (2. 3) and (2. 4) above is represented by

$$\psi(z, z^*) = \frac{1}{2i} \int_{\Gamma} R \frac{\partial \psi}{\partial n} |dt|$$

where n is the outward normal to Γ and $|dt|$ is arc length. The result, without the detailed proof, is given in [10].

2. Convergent series representation of the edge cusp

An axial cross section of figure 1 yields figure 3, which is composed of four symmetric curves. With the origin of the z-plane at the center of this figure let any one of these four curves be Γ which intersects the y-axis at z_1. That portion of Γ lying in an arbitrarily small neighborhood of the edge cusp can be written as a convergent series.

$$\bar{z} = g(z) = -z + iC(z - z_1)^{3/2} + \sum_{n=4}^{\infty} a_n (z - z_1)^{\frac{n}{2}}$$

This was proved in section 5 of this dissertation. The representation is valid for each of the four curves constituting the upper and lower cusps of figure 3.

3. Asymptotic representation of the point cusp

Consider the curve of paragraph 3 above which intersects the x-axis

at z_0. In a small neighborhood of z_0, the curve Γ is represented asymptotically by

$$\bar{z} = g(z) = z - a_0 i(z - z_0)^{3/2} \left\{ \frac{-1}{\ln(z - z_0)} - \frac{2\ln(-\ln(z - z_0))}{(\ln(z - z_0))^2} + \frac{\frac{-9}{4} + 2\ln(\frac{a_0}{4})}{(\ln(z - z_0))^2} + 0\left(\frac{(\ln(-\ln(z - z_0)))^2}{(\ln(z - z_0))^3}\right) \right\}$$

This was proved in section 7 of this dissertation. The representation is valid for each half of both the right and left cusps of figure 3.

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